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The Polytopes with Regular-Prismatic Vertex Figures

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IX. *The Polytopes with Regular-Prismatic Vertex Figures.*

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Preface.

The regular polytopes in two and three dimensions (polygons and polyhedra) and the "Archimedean solids" have been known since ancient times. To these, KEPLER and POINSOT added the regular star-polyhedra.

About the middle of last century, L. SCHLÄFLI* discovered the (convex) regular polytopes in more dimensions. As he was ignorant of two of the four KEPLER-POINSOT polyhedra, his enumeration of the analogous star-polytopes in four dimensions remained to be completed by E. HESS.†

Recently, D. M. Y. SOMMERVILLE‡ interpreted the (convex) regular polytopes as partitions of elliptic space, and considered the analogous partitions of hyperbolic space.

Some particular processes, for constructing "uniform" polytopes analogous to the Archimedean solids, were discovered by Mrs. BOOLE STOTT§ and discussed in great detail (with the help of co-ordinates) by Prof. SCHOUTE.|| Further, E. L. ELTE¶ completely enumerated all the uniform polytopes having a certain "degree of regularity," these including seven new ones (in six, seven and eight dimensions).

The object of the present paper is to exhibit these seven polytopes (*here named* 2_{21}, 1_{22}, 3_{21}, 2_{31}, 1_{32}, 4_{21}, 2_{41}), *along with certain others, as members of one family; and to investigate the relevant groups of symmetries.*

I should like to express here my thanks to Prof. BAKER for his advice and encouragement. The Appendix in regard to the cubic surface, was suggested by him.

* "Réduction d'une Intégrale Multiple qui comprend l'arc du cercle et l'aire du triangle sphérique comme cas particuliers," 'LIOUVILLE's Journal,' vol. 20, p. 361 (1855).

† "Einleitung in die Lehre von der Kugelteilung," 'Marburg. Ber.,' p. 31 (1885).

‡ "The regular divisions of space of *n* dimensions and their metrical constants," 'Palermo Rendiconti,' vol. 48, p. 9 (1923).

§ "Geometrical deduction of semiregular from regular polytopes and space fillings," 'Amsterdam Proceedings (Koninklijke Akademie van Wetenschappen),' vol. 11, No. 1 (1913).

|| "Analytical treatment of the polytopes regularly derived from the regular polytopes," *ibid.*, vol. 11, Nos. 3, 5; vol. 12, No. 2.

¶ "The Semiregular Polytopes of the Hyperspaces," Hoitsema, Groningen, 1912.

1. Introduction.

1.1. A " polytope " is the appropriate extension, to many-dimensional space, of the familiar polygon and polyhedron, which we shall call Π_2 and Π_3 respectively. Π_1 is taken to mean a segment of a straight line, considered as bounded by its two end points Π_0.

An m-dimensional polytope Π_m is then defined inductively, as a simply-connected portion of m-space, bounded by a number (more than m) of polytopes Π_{m-1}, such that every Π_{m-2} (occurring among the boundaries of the Π_{m-1}'s) belongs to just two of the Π_{m-1}'s. (The number of Π_{m-1}'s or Π_{m-2}'s to which any Π_{m-3} belongs is, of course, more than two.) As we shall consider none but " convex " polytopes, it is assumed that no two Π_{m-1}'s have any common points not on their boundaries.

1.2. It follows from this definition that Π_m possesses " elements " Π_r for all values of r from 0 to $m-1$, and we can say it possesses one element Π_m, namely itself.

Let

$$({}^r|_m)$$

denote the number of elements Π_r; so that

$$({}^0|_m),\quad ({}^1|_m)\quad \text{and}\quad ({}^{m-1}|_m)$$

are the numbers of " vertices," " edges " and " bounding figures," respectively, and

1.21 $$({}^m|_m) = 1.$$

The special property which distinguishes the general polytope from other kinds of " configuration " is

1.22 $$\sum_{r=0}^{m} (-1)^r\,({}^r|_m) = 1.$$

When $m = 0$, this is a particular case of 1.21. When $m = 1$ and 2 it gives

1.23 $$({}^0|_1) = 2$$

(" a line has two ends ") and

1.24 $$({}^0|_2) = ({}^1|_2)$$

(" a polygon has as many sides as vertices "). When $m = 3$, it is the famous EULER's theorem, of which the neatest proof is LEGENDRE's* (by means of spherical triangles). For a proof in the general case, see § 16 of H. POINCARÉ's " Analysis Situs."†

We can replace m by any smaller number s in a general identity such as 1.21 or 1.22, if we let

$$({}^{r}|_{s}) \qquad (r \leq s \leq m)$$

denote the number of elements Π_r belonging to a particular element Π_s. It is convenient also to let

$$({}^{s}_{r}|_{m}) \qquad (r \leq s \leq m)$$

denote the number of elements Π_s to which a particular element Π_r belongs. If we restrict the Π_s's and Π_r to belong to one Π_n, the number of Π_s's is naturally called

$$({}^{s}_{r}|_{n}) \qquad (r \leq s \leq n \leq m).$$

Clearly

1.25 $$({}^{n}_{r}|_{n}) = 1 = ({}^{r}_{r}|_{n}).$$

1.3. A "sphere-analogue" is the locus of points (in m dimensions) at a fixed distance (called the " radius ") from a fixed point called the " centre." If any straight line is drawn through a fixed point P to meet a fixed sphere-analogue in A and B, it can be proved that the harmonic conjugate of P with respect to A and B lies in a fixed " prime " or $(m - 1)$-space. P and the prime are said to be " pole and polar " with respect to the sphere-analogue.

Two polytopes are said to be " reciprocals " (of one another) if the vertices of one and the primes (containing the bounding figures) of the other are poles and polars with respect to a sphere-analogue whose centre is strictly inside the polytopes. Thus we can say that a vertex Π_0 " reciprocates " into a bounding figure Π_{m-1}; that an edge Π_m the join of two vertices, reciprocates into an element Π_{m-2}, the common part of two bounding figures; and, generally, that reciprocally corresponding elements are of dimensions adding up to $m - 1$. Since a polytope is supposed to have one element Π_m to which all its lower elements belong, it is natural to assume that the reciprocal polytope—and therefore any polytope—possesses one hypothetical element Π_{-1} which belongs at the same time to every proper element Π_s. Expressing this idea numerically:

1.31 $$({}^{-1}|_{m}) = 1,$$

1.32 $$({}^{-1}|_{s}) = 1,$$

1.33 $$({}^{s}_{-1}|_{m}) = ({}^{s}|_{m}),$$

1.34 $$({}^{s}_{-1}|_{n}) = ({}^{s}|_{n}).$$

* " Éléments de Géométrie," liv. 7, Prop. 25 (1794).

† 'Journal de l'École Polytechnique,' vol. 1, p. 100 (1895).

If

1.35 $$s + s' = r + r' = n + n' = m - 1,$$

the following pairs of properties are reciprocal (*i.e.*, are equal, when regarded as corresponding properties of reciprocal polytopes):

1.351 $$({}^{s}|{}_{m}) \quad \text{and} \quad ({}^{s'}|{}_{m}),$$

1.352 $$({}^{s}|{}_{n}) \quad \text{and} \quad ({}^{s'}_{n'}|{}_{m}),$$

1.353 $$({}^{s}_{r}|{}_{n}) \quad \text{and} \quad ({}^{s'}_{n'}|{}_{r'}),$$

1.34 exhibits 1.351 and 1.352 as special cases of 1.353.

By means of this rule, every identity can be reciprocated to give another identity Thus 1.21 gives 1.31, while 1.23 and 1.24 give respectively

1.36 $$({}^{m-1}_{m-2}|{}_{m}) = 2$$

and

1.37 $$({}^{m-1}_{m-3}|{}_{m}) = ({}^{m-2}_{m-3}|{}_{m}).$$

(These two identities can be generalized by changing m into n throughout.) 1.31 allows 1.22 to be written in the self-reciprocal form

1.38 $$\sum_{r=-1}^{m} (-1)^{r} ({}^{r}|{}_{m}) = 0.$$

1.4. If Π_m possesses $({}^{r}|{}^{\rho}_{m})$ Π_r's of a special type ρ, and $({}^{s}|{}^{\sigma}_{m})$ Π_s's of type σ; such that every Π_r of type ρ belongs to $({}^{s}_{r}|{}^{\sigma\rho}_{m})$ Π_s's of type σ, while every Π_s of type σ possesses $({}^{r}|{}^{\rho\sigma}_{s})$ Π_r's of type ρ; then, after a little consideration, it is seen that

1.41 $$({}^{r}|{}^{\rho}_{m})({}^{s}_{r}|{}^{\sigma\rho}_{m}) = ({}^{r}|{}^{\rho\sigma}_{s})({}^{s}|{}^{\sigma}_{m}) \qquad (r \leq s \leq m).$$

Here the "ρ" and "σ" can be omitted (independently) if all Π_r's or all Π_s's are of one type.

Changing m into n, reciprocating according to 1.352 and 1.353, and dropping the dashes, we obtain the more general theorem

1.42 $$({}^{r}_{n}|{}^{\rho}_{m})({}^{s}_{n}|{}^{\sigma\rho}_{r}) = ({}^{r}_{s}|{}^{\rho\sigma}_{m})({}^{s}_{n}|{}^{\sigma}_{m}) \qquad (n \leq s \leq r \leq m),$$

whose meaning should by this time be clear without detailed explanation of the symbols. 1.41 (with r and s interchanged) can be obtained from 1.42 by putting $n = -1$.

1.5. So far, we have tacitly assumed the polytope to be finite. But it is convenient to regard an infinite set of finite polytopes Π_{m-1}, fitting together to fill an $(m-1)$-space, as a "degenerate polytope" in m dimensions, the Π_{m-1}'s being its "bounding figures." All the properties

$$({}^{r}|_{m}) \qquad (0 \leq r \leq m-1)$$

are now infinite, but can be regarded as tending to infinity in definite mutual ratios, the ratio $({}^{r}|^{\rho}_{m}) : ({}^{s}|^{\sigma}_{m})$ being given by 1.41. Selecting then a set of finite numbers $({}^{r}|_{m})'$ having these proper ratios, 1.22 becomes

1.51
$$\sum_{r=0}^{m-1} (-1)^{r} ({}^{r}|_{m})' = 0.$$

To take a very simple example, "squared paper," regarded as a degenerate polyhedron bounded by squares, has an infinity of vertices, edges and faces, but we can still say

$$({}^{0}|_{3}) : ({}^{1}|_{3}) : ({}^{2}|_{3}) = 1 : 2 : 1,$$

and these numbers satisfy

$$1 - 2 + 1 = 0.$$

Degenerate polytopes, like finite ones, occur in reciprocal pairs; but now, of course, there are no sphere-analogues or harmonic ranges to help us. The rule given in 1.35 for reciprocally corresponding elements still applies, if we obtain the reciprocal of a given degenerate polytope by taking, for vertices, *any* points inside the original bounding figures, and joining them up suitably. The identity 1.51 is self-reciprocal.

1.6. The operation of moving or reflecting any polytope (preserving all distances among its component parts), in such a way as to leave it unchanged as a whole, is called a "symmetry" of the polytope. The totality of symmetries (including identity) of any given polytope Π_m forms a *group*, whose order will be called g_m.

If the symmetries of Π_m suffice to change (in turn) every one of a certain set of Π_r's into a particular Π_r of the set, these Π_r's are said to be "equivalent." (Clearly, equivalent elements must be equal.)

1.7. We are now in a position to give an inductive definition of "uniform polytope." Π_0 and Π_1 are supposed to be "uniform" always. As a basis for the induction, a polygon Π_2 is said to be uniform if it is "regular," *i.e.*, if its sides are equal and its vertices concyclic. Finally, a polytope in more than two dimensions is said to be uniform if its bounding figures are uniform and its vertices equivalent.

From now on, " Π_m " will always mean a uniform polytope. Since the symmetries permute the vertices, which are equivalent, we have

1.71 $$({}^0|_m) \leq g_m \leq ({}^0|_m)!\,.$$

It follows that g_m is finite or infinite according as Π_m is finite or degenerate.

As three-dimensional examples: the equilateral-triangular right prism with height equal to side, is a finite uniform polyhedron; and the plane filled up with alternate infinite strips of squares and of equilateral triangles, is a degenerate one.

It is easily proved by induction that all the elements of a uniform polytope are uniform. In particular, all the two-dimensional elements are regular polygons. It can also be proved by induction that the edges of a uniform polytope are all equal. Their common length will usually be taken as unity. A polytope of edge a similar to Π_m will be called $\Pi_m a$, or, if a is unspecified, $\Pi_m \times$.

1.8. We shall assume that the vertices of a finite uniform polytope, being a finite set of equivalent points, necessarily lie on a sphere-analogue, whose centre (called the " centre " of the polytope) is invariant for all symmetries. The radius of this " circum-scribing sphere-analogue " is called the " circum-radius " of the polytope. The $(m-1)$-space filled by a degenerate polytope may be regarded as a limiting kind of circum-scribing sphere-analogue, with its centre at infinity in the normal direction.

For reciprocating a finite uniform polytope, we shall always use a *concentric* sphere-analogue. (The *shape* of the reciprocal polytope is, of course, independent of the *size* of the reciprocating sphere-analogue.) In order to reciprocate a degenerate uniform polytope, we shall always take for vertices the *centres* of the original bounding figures. The reciprocal of a uniform polytope is not in general uniform; but it obviously has precisely the same symmetries, and therefore equivalent elements reciprocate into equivalent elements.

1.9. If all the elements Π_r are equivalent, for each r less than some number l, while the elements Π_l are not all equivalent; it is convenient to give Π_m special names, for the larger values of l: Π_m is said to be " super-Archimedean," " Archimedean " or " sub-Archimedean " if $l = m-1$, $m-2$ or $m-3$, respectively. The Archimedean polytopes are further sub-divided into " pure," " isohedral " and " mixed," Archimedean polytopes: " pure " if the Π_{m-2}'s (though not equivalent) are equal, and otherwise " isohedral " or " mixed " according as the Π_{m-1}'s are, or are not, all equal.

The ordinary " Archimedean solids " belong to the " super-Archimedean " and " pure Archimedean " categories.

2. *Vertex Figures.*

2.1. The definition 1.7 may seem somewhat artificial. It was devised in order that a uniform polytope, so defined, should be uniquely determined (in shape) by the neigh-bourhood of one vertex, *i.e.*, by what happens inside an arbitrarily small sphere-analogue drawn round one vertex. We shall assume then that, given any uniform polytope, there is no other uniform polytope of different shape having the same vertex neighbourhood. J. C. P. MILLER has made the interesting discovery of a non-uniform polytope (in three dimensions, bounded by 8 triangles and $8 + 8 + 2$ squares) whose vertex neighbourhood is unique and the same as that of the uniform " small rhombicuboctahedron."*

It is desirable to define some sort of indicatrix which will give a clear idea of the vertex neighbourhood of a uniform polytope. The vertex neighbourhood (*i.e.*, vertex angle) of a regular k-gon is determined by the distance, $2 \cos \pi/k$, between points measured off at unit distances along two covertical sides. We say that a line of length $2 \cos \pi/k$ is the " vertex figure " of the polygon. This idea can be extended to more dimensions.

2.2. First suppose that the given uniform polytope Π_m is of *unit* edge. Those vertices which are the further ends of all edges at a particular vertex A, then lie on the sphere-analogue of unit radius, centre A, as well as on the circumscribing sphere-analogue (1.8) of the whole polytope. Being on the intersection of two sphere-analogues, these vertices lie in a prime ϖ, and are therefore the complete set of vertices of an $(m - 1)$-dimensional polytope $\Pi_{m-1,\,1}$. This polytope is called the " vertex figure " of Π_m. (It is generally not of unit edge, nor even uniform.) Its $(s - 1)$-dimensional elements $\Pi_{s-1,\,1}$ may be seen to be the vertex figures of those s-dimensional elements of Π_m which occur at A. In particular, corresponding to any k-gonal Π_2's of Π_m, $\Pi_{m-1,\,1}$ has edges of length $2 \cos \dfrac{\pi}{k}$.

The vertex figure of a degenerate uniform polytope has unit circum-radius, since its centre is A. The centre of the vertex figure of a finite uniform polytope is the intersection of the prime ϖ with the line joining A to the centre of the polytope.

The vertex figure is independent of the choice of A, since all vertices are equivalent.

2.3. In order that similar polytopes may have identical vertex figures, we must define the vertex figure of a uniform polytope of *arbitrary* edge length as having for vertices points measured off at *unit* distances along a set of covertical edges. The figure so obtained is clearly similar to that determined by the *ends* of the edges.

In virtue of this definition, the assumption at the beginning of 2.1 implies that two uniform polytopes with the same vertex figure must be similar. In nearly all cases the similarity is " direct " (*i.e.*, the two polytopes can be superposed by means of shrinkage and motion in their own space). But the " snub cube "* (KEPLER's " Cubus Simus ") and " snub dodecahedron "* both exist in two enantiomorphous varieties (which cannot be superposed without reflection or four-dimensional motion). In each of these cases, the two varieties have the same vertex figure; *e.g.*, the lævo- and dextro-snub cube, each bounded by $8 + 24$ triangles and 6 squares, both have four triangles and

* 'Encyclopædia Britannica,' 11th edition, art. " Polyhedron."

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one square at each vertex, so that the vertex figure of either solid is a cyclic pentagon of sides

$$1,\ 1,\ 1,\ 1,\ \sqrt{2}$$

(which is unique). No such exceptional polytopes have been found in more dimensions.

2.4. It is evident that all those symmetries of a uniform polytope which leave one vertex invariant occur as symmetries of the vertex figure. It is generally true that they include *all* the symmetries of the vertex figure. We shall assume that the only cases of failure of this theorem are those provided by the two "snub solids" (2.3), whose vertex figures (being cyclic pentagons with four equal sides) have a reflective symmetry not shared by the whole polyhedron. With these two exceptions then, if $g_{m-1,1}$ denotes the order of the group of symmetries of $\Pi_{m-1,1}$,

2.41 $$g_m = ({}^{0}|{}_{m})\, g_{m-1,1}.$$

For the snub solids, on the other hand

2.42 $$g_m = ({}^{0}|{}_{m})$$

although $g_{m-1,1} = 2$.

2.5. Let

$$({}^{s-1}|{}_{m-1,1})$$

denote the number of $(s-1)$-dimensional elements $\Pi_{s-1,1}$ possessed by the vertex figure $\Pi_{m-1,1}$. We have seen (2.2) that these elements simply correspond to the Π_s's at one vertex of Π_m. Hence

2.51 $$({}^{s-1}|{}_{m-1,1}) = ({}^{s}_{0}|{}_{m}).$$

Substituting in **1.41**, with $r = 0$, we have

$$({}^{0}|{}_{m})\,({}^{s-1}|{}^{\sigma}_{m-1,1}) = ({}^{0}|{}^{\sigma}_{s})\,({}^{s}|{}^{\sigma}_{m}),$$

i.e.,

2.52 $$({}^{s}|{}^{\sigma}_{m}) = ({}^{0}|{}_{m})\,({}^{s-1}|{}^{\sigma}_{m-1,1})/({}^{0}|{}^{\sigma}_{s}).$$

If we know the properties of the vertex figure of Π_m and the number of vertices of Π_m, we can thus obtain the number of Π_s's which have $({}^{0}|{}^{\sigma}_{s})$ vertices.

Since the vertices of a polytope correspond to the bounding figures of its reciprocal, the reciprocal of Π_m has only one kind of bounding figure, and this bounding figure is the reciprocal of $\Pi_{m-1,1}$ (with respect to an $(m-1)$-dimensional sphere-analogue concentric with $\Pi_{m-1,1}$). This agrees with the fact that (by 1.35) $({}^{s}|{}_{m-1})$ and $({}^{s'}|{}_{m-1,1})$ are reciprocal properties if $s + s' = m - 2$.

2.6. If $\Pi_{m-1,1}$ happens to be uniform, its vertex figure is denoted by $\Pi_{m-2,2}$ and is called the "second vertex figure" of Π_m. Extending this idea: *if* $\Pi_{m-u+1,u-1}$ *is uniform,* its vertex figure $\Pi_{m-u,u}$ is called the "uth vertex figure" of Π_m. It follows that $\Pi_{m-u,u}$ is the $(u-v)$th vertex figure of $\Pi_{m-v,v}$, and that the uniformity of $\Pi_{m-u+1,u-1}$ implies the existence of $\Pi_{m-v,v}$ for all $v \leq u$. $\Pi_{m,0}$ must be taken to mean Π_m. The existence of $\Pi_{1,m-1}$ (*i.e.*, the regularity of $\Pi_{2,m-2}$) would trivially imply the existence of $\Pi_{0,m}$.

The $(s-u)$-dimensional elements $\Pi_{s-u,u}$ of $\Pi_{m-u,u}$ may be seen to be the uth vertex figures of those s-dimensional elements of Π_m which occur at one Π_{u-1}. Expressing this fact numerically (the properties of $\Pi_{p,u}$ being distinguished from those of Π_p by changing " p " into " p, u "), we have

2.61 $$\left({}^{s-u}|_{m-u,u}\right) = \left({}^{s}_{u-1}|_m\right) \quad (u-1 \leq s \leq m).$$

This is a special case of the still more obvious relation

2.62 $$\left({}^{s-u}_{r-u}|_{m-u,u}\right) = \left({}^{s}_{r}|_m\right) \quad (u-1 \leq r \leq s \leq m),$$

in which we may, as usual, replace m by n $(\leq m)$.

Substituting 2.61 in 1.41, with $r = u-1$, we have

2.63 $$\left({}^{u-1}|_m\right)\left({}^{s-u}|_{m-u,u}^{\sigma}\right) = \left({}^{u-1}|^{\sigma}_{s}\right)\left({}^{s}|^{\sigma}_{m}\right) \quad (u-1 \leq s \leq m).$$

(The assumption that $\Pi_{m-u,u}$ exists implies that the Π_{u-1}'s are all of one type.)

If Π_m has a uth vertex figure, 2.41 can be extended so as to give

2.64 $$g_m = \left({}^{0}|_m\right)\left({}^{0}|_{m-1,1}\right)\left({}^{0}|_{m-2,2}\right)\cdots\left({}^{0}|_{m-u+1,u-1}\right) g_{m-u,u}.$$

2.7. Let
$$_r\mathrm{R}_m \quad (r \leq m)$$

denote the central distance of a Π_r, *i.e.*, the distance from the centre of Π_m to the centre of one of its elements Π_r; so that, in particular, $_0\mathrm{R}_m$ denotes the circum-radius of Π_m. Analogously, let
$$_r\mathrm{R}_n \quad (r \leq n)$$

denote the distance from the centre of a Π_n to the centre of a Π_r belonging to the Π_n.

Since the line joining the centre of a sphere-analogue to the centre of the section by an n-space is perpendicular to the space of section, $_n\mathrm{R}_m$ and $_r\mathrm{R}_n$ and $_r\mathrm{R}_m$ must form a right-angled triangle. So

2.71 $$({}_r\mathrm{R}_m)^2 = ({}_r\mathrm{R}_n)^2 + ({}_n\mathrm{R}_m)^2 \quad (r \leq n \leq m).$$

In particular (putting $r = n$)

2.72 $$_n\mathrm{R}_n = 0$$

and (putting $r = 0$)

2.73 $$({}_n\mathrm{R}_m)^2 = ({}_0\mathrm{R}_m)^2 - ({}_0\mathrm{R}_n)^2.$$

2.8. Let $2\theta_m$ denote the angle subtended at the centre of Π_m by an edge, and $2\theta_{p,u}$ the corresponding property of $\Pi_{p,u}$. Then, supposing Π_m to be of unit edge,

2.81 $$_0\mathrm{R}_m = \tfrac{1}{2}\operatorname{cosec}\theta_m.$$

If Π_m has a k-gonal Π_2, $\Pi_{m-1,1}$ has an edge of length $2\cos\theta_2$ where $\theta_2 = \pi/k$. So, if $\theta_{m-1,1}$ refers to this edge, the circum-radius of $\Pi_{m-1,1}$ is given by

2.82. $${}_0R_{m-1,1} = \cos\theta_2 \operatorname{cosec}\theta_{m-1,1}.$$

A glance at the diagram (Fig. 1; in which O is the centre of Π_m, AB an edge, and Q the centre of the actual vertex figure at A) reveals the fact that

2.83 $${}_0R_{m-1,1} = \cos\theta_m.$$

(Thus a given polytope $\Pi_{m-1,1}$ *cannot* be the vertex figure of a real polytope if

2.84 $${}_0R_{m-1,1} > 1.)$$

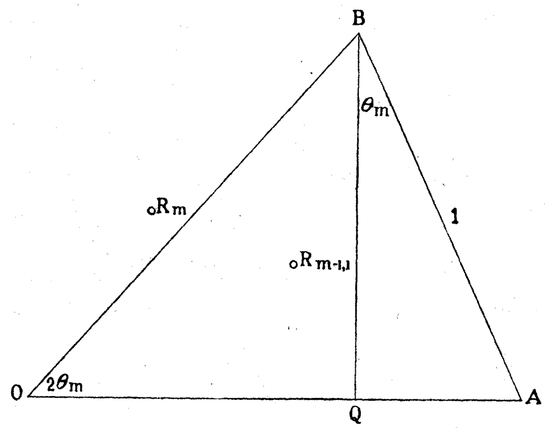

FIG. 1.

Combining 2.83 and 2.82,

2.85 $$\cos\theta_m = \cos\theta_2 \operatorname{cosec}\theta_{m-1,1}$$

and so

2.86 $$\sin^2\theta_m = 1 - \frac{\cos^2\theta_2}{\sin^2\theta_{m-1,1}}.$$

If Π_m has a uth vertex figure, we may apply 2.86 to $\Pi_{m-u+1,u-1}$, obtaining

2.861 $$\sin^2\theta_{m-u+1,u-1} = 1 - \frac{\cos^2\theta_{2,u-1}}{\sin^2\theta_{m-u,u}}.$$

Therefore $\sin^2\theta_m$ can be expressed as a continued fraction*:

2.87 $$\sin^2\theta_m = 1 - \frac{\cos^2\theta_2}{1-}\,\frac{\cos^2\theta_{2,1}}{1-}\,\frac{\cos^2\theta_{2,2}}{1-}\cdots\frac{\cos^2\theta_{2,u-1}}{1-\cos^2\theta_{m-u,u}}.$$

In particular, if Π_m has an $(m-2)$th vertex figure,

$$\sin^2\theta_m = 1 - \frac{\cos^2\theta_2}{1-}\,\frac{\cos^2\theta_{2,1}}{1-}\,\frac{\cos^2\theta_{2,2}}{1-}\cdots\frac{\cos^2\theta_{2,m-3}}{1-\cos^2\theta_{2,m-2}}$$

2.88 $$= \Delta_m/\Delta_{m-1,1}$$

where, in accordance with the algebra of continued fractions, Δ_m is defined by

2.89 $$\begin{cases} \Delta_1 = 1, \\ \Delta_2 = \sin^2\theta_2, \\ \Delta_{v+2} = \Delta_{v+1} - \Delta_v\cos^2\theta_{2,v}, \end{cases}$$

and $\Delta_{m-u,u}$ is obtained from Δ_{m-u} by changing θ_2 into $\theta_{2,u}$ and $\theta_{2,v}$ into $\theta_{2,v+u}$.

* This use of continued fractions is due to SCHLÄFLI.

2.9. Applying 2.88 to $\Pi_{m-u,u}$ (still supposing the existence of $\Pi_{2,m-2}$),

$$\sin^2 \theta_{m-u,u} = \Delta_{m-u,u}/\Delta_{m-u-1,u+1} \qquad (u \leq m-3).$$

Also

$$\sin^2 \theta_{2,m-2} = \Delta_{2,m-2}.$$

Hence

2.91 $$\sin^2 \theta_m \; \sin^2 \theta_{m-1,1} \; \sin^2 \theta_{m-2,2} \; \ldots \; \sin^2 \theta_{2,m-2} = \Delta_m,$$

and therefore

2.92 $$\Delta_m \geq 0.$$

Combining 2.81 and 2.88,

2.93 $${}_0R_m = \tfrac{1}{2}(\Delta_{m-1,1}/\Delta_m)^{\frac{1}{2}}.$$

Thus the polytope is degenerate if

2.94 $$\Delta_m = 0.*$$

The recurrence formula 2.89 enables Δ_m to be expressed as an m-row determinant,† namely

2.95 $$\Delta_m = \begin{vmatrix} 1 & \cos\theta_2 & 0 & 0 & 0 & \cdots & 0 \\ \cos\theta_2 & 1 & \cos\theta_{2,1} & 0 & 0 & \cdots & 0 \\ 0 & \cos\theta_{2,1} & 1 & \cos\theta_{2,2} & 0 & \cdots & 0 \\ \vdots & & & & & & \vdots \\ 0 & \cdots & 0 & \cos\theta_{2,m-4} & 1 & \cos\theta_{2,m-3} & 0 \\ 0 & \cdots & 0 & 0 & \cos\theta_{2,m-3} & 1 & \cos\theta_{2,m-2} \\ 0 & \cdots & 0 & 0 & 0 & \cos\theta_{2,m-2} & 1 \end{vmatrix}.$$

3. *Regular Polytopes.*

3.1. The very simple polytopes Π_0 and Π_1 are supposed to be automatically "regular." A regular polygon has already been defined (1.7). We define a "regular polytope" inductively as a uniform polytope whose vertex figure is regular. This definition is exactly equivalent to saying that an m-dimensional polytope is "regular" if it has an mth vertex figure. Thus Π_m, if regular, possesses the complete set of vertex figures $\Pi_{m-u,u}$, from $\Pi_{m,0}$ ($= \Pi_m$) down to $\Pi_{0,m}$ (a mere point), and all these are regular. In particular, since there is an $(m-2)$th vertex figure, 2.9 is relevant.

By 2.2, the vertex figure of a bounding figure of Π_m is a bounding figure of the vertex figure of Π_m. This principle enables us to prove by induction (through the series of vertex figures) that the bounding figures of a regular polytope are regular, thence that

* Polytopes for which $\Delta_{m-1,1} = 0$ are "improper," since they require $\cos\theta_2 \cos\theta_{2,1} \cos\theta_{2,2} \ldots = 0$. For by 2.91, $\Delta_{m-1,1} = 0$ implies $\sin\theta_{m-u,u} = 0$ for some $u > 0$, while, by 2.861, $\sin\theta_{m-u,u} = 0$ $(u > 0)$ implies $\cos\theta_{2,u-1} = 0$.

† *Cf.* SCHLÄFLI, *loc. cit.*, § VI.

all the elements are regular, and finally that everything of the form $\Pi_{p,u}$ (which can be regarded either as a p-dimensional element of $\Pi_{m-u,\,u}$ or as the uth vertex figure of Π_{p+u}) is regular.

3.2. We proceed to prove by induction that the elements Π_n of a regular polytope Π_m are equivalent. As a basis for the induction, we know that the vertices of the nth vertex figure of Π_m are equivalent (since the nth vertex figure is regular and, *a fortiori*, uniform). Now suppose that the elements $\Pi_{n-1,\,1}$ of the vertex figure $\Pi_{m-1,\,1}$ are equivalent.

Take any two Π_n's of Π_m. Since the vertices of Π_m are equivalent, there exists a symmetry which will change the first Π_n into another Π_n having one vertex (A, say) in common with the second Π_n. We thus obtain two Π_n's with a common vertex A. By 2.2, their actual vertex figures at A are elements $\Pi_{n-1,\,1}$ of the actual vertex figure of Π_m at A. But we are supposing the elements $\Pi_{n-1,\,1}$ of $\Pi_{m-1,\,1}$ to be equivalent. Hence there exists a symmetry of $\Pi_{m-1,\,1}$ which will change one of these two $\Pi_{n-1,\,1}$'s into the other. By the assumption in 2.4 (since the exceptional snub solids are not regular), this symmetry, regarded as leaving A invariant, is a symmetry of Π_m. As such, it must change one of the two Π_n's at A into the other ; for, otherwise, Π_m would possess two different Π_n's having the vertex A and their vertex figure at A in common, which is absurd.

The combination or " product " of the two symmetries here described establishes the equivalence of the original (arbitrarily chosen) pair of Π_n's, and hence the equivalence of all the Π_n's. *A fortiori*, all the Π_n's are equal. Thus we can speak of *the* Π_n, and so also of *the* $\Pi_{n-u,\,u}$.

Throughout 3.2, we have really assumed, concerning Π_m, nothing more than that its nth vertex figure is uniform. We can therefore assert the following more general theorem :

If Π_m has a uth vertex figure, then for all n (strictly) less than u,

3.21 the Π_{n+2}'s are regular,

3.22 the Π_{n+1}'s are equal,

3.23 the Π_n's are equivalent.

(3.21 follows from the uniformity of the nth vertex figure, which implies the regularity of $\Pi_{2,\,n}$. 3.22 follows from 3.21, since unequal Π_{n+1}'s would somewhere have to belong to the same Π_{n+2}.)

3.3. We shall next prove that the reciprocal of a regular polytope is regular. This is trivially true in one dimension. Suppose it true for every regular polytope in $m-1$ dimensions.

Consider any regular polytope Π_m in m dimensions, and let Π'_m be its reciprocal. The bounding figure of Π'_m, being (by 2.5) reciprocal to the (regular) vertex figure of

Π_m, is regular (by hypothesis). Also, since equivalent elements reciprocate into equivalent elements (1.8), the vertices of Π'_m, which correspond to the bounding figures of Π_m, are equivalent (3.2). Hence Π'_m is uniform (1.7). Its vertex figure, being reciprocal to the bounding figure of Π_m, is regular. Hence Π'_m is regular (3.1).

3.31. Since the bounding figure of Π'_m is reciprocal to the vertex figure of Π_m, it follows (by induction) that the $(m-u)$-dimensional element of Π'_m is reciprocal to $\Pi_{m-u,\,u}$.

3.32. If Π_m is uniform and has an $(n+1)$th vertex figure, so that its Π_{n+1}'s are equal (3.22) and regular (3.21), then Π'_n will always be taken to mean the vertex figure of the reciprocal of Π_{n+1}. Π'_n has thus a definite edge-length, instead of being merely the reciprocal of Π_n (irrespective of size).

3.4. Schläfli* devised the following ingenious notation for regular polytopes. The regular polygon of k sides is called

$$\{k\}.$$

The regular polyhedron whose bounding figure and vertex figure are respectively

$$\{k_1\} \quad \text{and} \quad \{k_2\}\times$$

is called

$$\{k_1,\ k_2\}\ ;$$

and, generally, the regular polytope whose bounding figure and vertex figure are respectively

$$\{k_1,\ k_2,\ \dots\ k_{m-2}\} \quad \text{and} \quad \{k_2,\ \dots\ k_{m-2},\ k_{m-1}\}\times$$

is called

$$\{k_1,\ k_2,\ \dots\ k_{m-2},\ k_{m-1}\}.$$

The occurrence of "$k_2,\ \dots\ k_{m-2}$," in both the bounding figure and the vertex figure, is justified by the principle (3.1) that the vertex figure of the bounding figure is the bounding figure of the vertex figure.

If

3.41 $$\Pi_m = \{k_1,\ k_2,\ \dots\ k_{m-2},\ k_{m-1}\},$$

it is easily seen that

3.42 $$\Pi_s = \{k_1,\ k_2,\ \dots\ k_{s-2},\ k_{s-1}\},$$

$$\Pi_{m-u,\,u} = \{k_{u+1},\ k_{u+2},\ \dots\ k_{m-2},\ k_{m-1}\}\times$$

and

3.43 $$\Pi_{s-u,\,u} = \{k_{u+1},\ k_{u+2},\ \dots\ k_{s-2},\ k_{s-1}\}\times.$$

In particular,

3.44 $$\Pi_{2,\,u} = \{k_{u+1}\}\times.$$

Hence

3.45 $$k_{u+1} = ({}^0|_{2,\,u}) = ({}^1|_{2,\,u}).$$

* See the Preface. Actually Schläfli used round brackets instead of curly ones. The same notation (without brackets or commas) was employed by Sommerville, and by Van Oss, 'Amsterdam Proceedings,' vol. 12, No. 1 (1915).

Since $\Pi_{s-u,u}$ is the vertex figure of $\Pi_{s-u+1,u-1}$, the edge of $\Pi_{s-u,u}$ must be the vertex figure of $\Pi_{2,u-1}$, *i.e.* of $\{k_u\}\times$. This edge is therefore (2.1) of length $2\cos\pi/k_u$, and **3.43** becomes more precisely

3.46 $$\Pi_{s-u,u} = \{k_{u+1}, k_{u+2}, \ldots k_{s-2}, k_{s-1}\}\, 2\cos\frac{\pi}{k_u}.$$

In particular,

3.47 $$\Pi_{m-1,1} = \{k_2, k_3, \ldots k_{m-2}, k_{m-1}\}\, 2\cos\frac{\pi}{k_1}.$$

The vertex figure being thus definite, **2.3** shows that two different regular polytopes cannot have the same Schläfli symbol. But it is only for certain special values of the k's that the polytope $\{k_1, k_2, \ldots k_{m-2}, k_{m-1}\}$ can exist at all. These special values will now be determined.

By the definition of $\theta_{p,u}$ in **2.8**,

3.48 $$\theta_{2,u} = \frac{\pi}{k_{u+1}}.$$

Hence, by **2.95**, Δ_m is a function of the k's. In particular—

3.49 $$\begin{cases} \Delta_1 = 1, \\ \Delta_2 = \sin^2\dfrac{\pi}{k_1}, \\ \Delta_3 = \sin^2\dfrac{\pi}{k_1} - \cos^2\dfrac{\pi}{k_2}, \\ \Delta_4 = \sin^2\dfrac{\pi}{k_1}\sin^2\dfrac{\pi}{k_3} - \cos^2\dfrac{\pi}{k_2}, \\ \Delta_5 = \left(\sin^2\dfrac{\pi}{k_1} - \cos^2\dfrac{\pi}{k_2}\right)\sin^2\dfrac{\pi}{k_4} - \sin^2\dfrac{\pi}{k_1}\cos^2\dfrac{\pi}{k_3}. \end{cases}$$

3.5. Supposing

$$k_u > 2$$

(since the "digon" $\{2\}$, which encloses no space, is not strictly a polytope according to 1.1), we shall enumerate all the regular polytopes which can be obtained from the following two *necessary* conditions :

3.51 $$\{k_1, k_2, \ldots k_{m-2}\} \quad \text{and} \quad \{k_2, \ldots k_{m-2}, k_{m-1}\}$$

exist and are finite ; and

3.52 $$k_1, k_2, \ldots k_{m-2}, k_{m-1}$$

satisfy $\Delta_m \geq 0$. (By **2.93**, the polytope is finite if $\Delta_m > 0$.)

It will appear later that these conditions are not only necessary but *sufficient.*

In one dimension, we admit Π_1 by writing

$$\Pi_1 = \{\}.$$

In two dimensions, $\{k_1\}$ is admitted for all k_1, the *degenerate* polygon $\{\infty\}$ being an infinite straight line broken into consecutive segments of unit length.

In three, four and five dimensions, we have :—

	(*Finite*)	(*Degenerate*)
$m = 3.$	{3, 3}, {3, 4}, {4, 3}, {3, 5}, {5, 3}	{4, 4}, {3, 6}, {6, 3}
$m = 4.$	{3, 3, 3}, {3, 3, 4}, {4, 3, 3}, {3, 3, 5}, {5, 3, 3}, {3, 4, 3}	{4, 3, 4}
$m = 5.$	{3, 3, 3, 3}, {3, 3, 3, 4}, {4, 3, 3, 3}	{4, 3, 3, 4}, {3, 3, 4, 3}, {3, 4, 3, 3}

Since {3, 3, 3, 3}, {3, 3, 3, 4} and {4, 3, 3, 3} are the only finite regular polytopes in five dimensions, it follows by repeated application of 3.51 that the only remaining possibilities ($m > 5$) are :—

3.53
$$\begin{cases} \alpha_m = \{3, 3, \ldots 3, 3\}, \\ \beta_m = \{3, 3, \ldots 3, 4\}, \\ \gamma_m = \{4, 3, \ldots 3, 3\}, \\ \delta_m = \{4, 3, \ldots 3, 4\}. \end{cases}$$

These all satisfy

$$\Delta_m (k_1, k_2, \ldots k_{m-2}, k_{m-1}) \geqq 0.$$

For, we can prove (by induction, using 2.89 in the form

$$\Delta_1 = 1, \qquad \Delta_2 = \sin^2 \frac{\pi}{k_1}, \qquad \Delta_{u+1} = \Delta_u - \Delta_{u-1} \cos^2 \frac{\pi}{k_u})$$

that

$$\Delta_m (3, 3, \ldots 3, 3) = (m+1)/2^m,$$

$$\Delta_m (3, 3, \ldots 3, 4) = \Delta_m (4, 3, \ldots 3, 3) = 1/2^{m-1}$$

and

$$\Delta_m (4, 3, \ldots 3, 4) = 0.$$

Thus α_m, β_m, γ_m are finite, while δ_m is degenerate.

Actually, α_m, β_m and γ_m are well known under the respective names "regular simplex," "cross polytope" and "measure polytope." In particular,

$\alpha_3 = \{3, 3\}$ is the regular tetrahedron,

$\beta_3 = \{3, 4\}$ is the octahedron,

$\gamma_3 = \{4, 3\}$ is the cube.

Also

$\delta_3 = \{4, 4\}$ is the "squared paper" pattern (1.5).

It follows at once from **3.53** that

$$\begin{cases} \alpha_m \text{ has bounding figure } \alpha_{m-1} \text{ and vertex figure } \alpha_{m-1}, \\ \beta_m \quad ,, \quad ,, \quad ,, \quad \alpha_{m-1} \quad ,, \quad ,, \quad ,, \quad \beta_{m-1}, \\ \gamma_m \quad ,, \quad ,, \quad ,, \quad \gamma_{m-1} \quad ,, \quad ,, \quad ,, \quad \alpha_{m-1}\sqrt{2}, \\ \delta_m \quad ,, \quad ,, \quad ,, \quad \gamma_{m-1} \quad ,, \quad ,, \quad ,, \quad \beta_{m-1}\sqrt{2}. \end{cases}$$

Using these facts to define α_m, β_m, γ_m, δ_m when m is small, we have successively :—

$$\alpha_2 = \{3\}, \qquad \alpha_1 = \{\,\} = \Pi_1 \text{ (unit length)}, \qquad \alpha_0 = \Pi_0,$$
$$\beta_2 = \{4\}, \qquad \beta_1 = \alpha_1\sqrt{2},$$
$$\gamma_2 = \beta_2, \qquad \gamma_1 = \alpha_1, \qquad \gamma_0 = \alpha_0,$$
$$\delta_2 = \{\infty\}.$$

Since the bounding figure and vertex figure of a regular polytope are reciprocal to the vertex figure and bounding figure of the reciprocal polytope, it is easily proved by induction that the polytopes

$$\{k_1, k_2, \ldots k_{m-2}, k_{m-1}\} \quad \text{and} \quad \{k_{m-1}, k_{m-2}, \ldots k_2, k_1\}$$

are reciprocal. In particular, β_m and γ_m are reciprocal, while α_m and δ_m are each self-reciprocal.

With the meaning assigned in **3.32**, we now have

3.54 $$\Pi'_n = \{k_{n-1}, k_{n-2}, \ldots k_2, k_1\}\, 2\cos\frac{\pi}{k_n}.$$

3.6. In order to prove that all these polytopes really exist, we shall specify Cartesian co-ordinates for all the vertices of each polytope (except the polygons, whose existence is obvious).

The notation here employed for co-ordinates is as follows :—

$$(x_1, x_2, \ldots x_n)$$

denotes the set of points obtained by permuting the x's in every possible way.

$$(x_1, x_2, \ldots x_n)'$$

denotes the set obtained by permuting them *evenly*. The sign of ambiguity ($\pm$) placed before a bracket indicates that every co-ordinate within may have either sign.

$$(x_1, \ldots x_p\,;\; x_{p+1}, \ldots x_q\,;\; x_{q+1}, \ldots)$$

denotes the set obtained by permuting $x_1, \ldots x_p$ among themselves, $x_{p+1}, \ldots x_q$ among themselves, and so on. In particular, $(x_1\,;\; x_2\,;\; \ldots)$ denotes a single point.

For degenerate polytopes, the co-ordinates are taken to be all integers (positive, zero and negative), arranged in every possible way, subject to whatever conditions are stated.

Sometimes (*e.g.*, in the case of $\alpha_m \sqrt{2}$) it is convenient to employ $m + 1$ co-ordinates with a constant sum (instead of simply m co-ordinates), in which case the polytope is to be regarded as lying in a prime of the $(m + 1)$-space.

τ always stands for the positive root of the equation $x^2 - x - 1 = 0$, so that

3.61 $$\tau = \tfrac{1}{2}(\sqrt{5} + 1) = 1 + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+} \dots \textit{ ad inf.}$$

On comparison with 3.5, it is seen that the following list contains co-ordinates (sometimes in two alternative forms) for the vertices of *all* the regular polytopes ($m > 2$).

$\alpha_m \sqrt{2}$:	$(1, 0, 0, \dots 0)$;	m zeros.
$\beta_m \sqrt{2}$:	$\pm (1, 0, 0, \dots 0)$;	$m - 1$ zeros.
$\gamma_m 2$:	$\pm (1, 1, 1, \dots 1)$;	m ones.
δ_m :	$(x_1, x_2, \dots x_{m-1})$	(all integers).
$\{3, 5\}\, 2$* :	$\pm (\tau, 1, 0)'$	(cyclically permuted).
$\{3, 5\}\, 2\tau^{-1}$:	$(\tau, 1, \tau^{-1}, 0)'$	(evenly permuted).
$\{5, 3\}\, 2\tau^{-1}$* :	$\pm (\tau, \tau^{-1}, 0)'$ $\pm (1, 1, 1)$	(together).
$\{5, 3\}\, 4\tau^{-1}$:	$(3, -1, -1, -1)$, $(\sqrt{5}, 1, -1, -\sqrt{5})'$, $(1, 1, 1, -3)$.	
$\{3, 6\}\, \sqrt{2}$:	(x_1, x_2, x_3) ;	$x_1 + x_2 + x_3 = 0$.
$\{6, 3\}\, \sqrt{2}$:	$(1, 0, -1)$ (mod. 3) ;	$x_1 + x_2 + x_3 = 0$.
$\{3, 3, 5\}\, 2\tau^{-1}$† :	$\pm (\tau, 1, \tau^{-1}, 0)'$, $\pm (2, 0, 0, 0)$, $\pm (1, 1, 1, 1)$.	
$\{3, 3, 5\}\, 2\sqrt{2}\tau^{-1}$:	$(\tau^2, \tau^{-1}, \tau^{-1}, \tau^{-1})$ $(\tau, \tau, \tau, \tau^{-2})$ $(\sqrt{5}, 1, 1, 1)$ $\pm (2, 2, 0, 0)$.	(1 or 3 minuses), (1 or 3 minuses), (0, 2 or 4 minuses),

* *Cf.* Schoute's "Analytical treatment of the polytopes ..." (*loc. cit.* in Preface), § 123.

† *Ibid.*, § 160.

$\{5, 3, 3\}\ 2\tau^{-2}$* : $\begin{cases} \pm (2, \tau, 1, \tau^{-1})', \\ \pm (\tau, \sqrt{5}, \tau^{-1}, 0)', \\ \pm (\tau^2, 1\ \tau^{-2}, 0)', \\ \pm (\tau^2, \tau^{-1}, \tau^{-1}, \tau^{-1}), \\ \pm (\tau, \tau, \tau, \tau^{-2}), \\ \pm (\sqrt{5}, 1, 1, 1), \\ \pm (2, 2, 0, 0). \end{cases}$

$\{3, 4, 3\}\ \sqrt{2}$† : $\pm (1, 1, 0, 0)$.

$\{3, 4, 3\}\ 2$† : $\begin{cases} \pm (2, 0, 0, 0), \\ \pm (1, 1, 1, 1). \end{cases}$

$\{3, 3, 4, 3\}\ \sqrt{2}$: (x_1, x_2, x_3, x_4); $x_1 + x_2 + x_3 + x_4 = 0$ (mod. 2).

$\{3, 3, 4, 3\}\ 2$: $\begin{cases} (0, 0, 0, 0) \text{ (mod. 2)}, \\ (1, 1, 1, 1) \text{ (mod. 2)}. \end{cases}$

$\{3, 4, 3, 3\}\ \sqrt{2}$: $(1, 1, 0, 0)$ (mod. 2).

3.7. Let g_m be the order of the group of symmetries of a regular polytope Π_m, and $g_{s-u,u}$ the corresponding property of $\Pi_{s-u,u}$. By 1.71,

$$g_0 = 1 \qquad (\text{so that } g_{0,u} = 1)$$

and

$$g_1 = 2 \qquad (\text{so that } g_{1,u} = 2).$$

By 2.41,

3.71 $$({}^0|_m) = g_m/g_{m-1,1}.$$

Similarly

$$({}^0|_s) = g_s/g_{s-1,1} \quad \text{and} \quad ({}^0|_{m-s,s}) = g_{m-s,s}/g_{m-s-1,s+1}.$$

Hence, by 2.52,

$$\begin{aligned}(g_s/g_m)\,({}^s|_m) &= (g_{s-1,1}/g_{m-1,1})\,({}^{s-1}|_{m-1,1}) \\ &= \dots \\ &= (g_{0,s}/g_{m-s,s})\,({}^0|_{m-s,s}) = 1/g_{m-s-1,s+1}\end{aligned}$$

and

3.72 $$({}^s|_m) = g_m/g_s\,g_{m-s-1,s+1}.$$

* *Ibid.*, § 160.

† *Ibid.*, § 144.

It follows that

$$({}^{s}|_{p,\,u}) = g_{p,\,u}/g_{s,\,u}\, g_{p-s-1,\,u+s+1}$$

and

$$({}^{s}_{r}|_{n}) = ({}^{s-r-1}|_{n-r-1,\,r+1}) = g_{n-r-1,\,r+1}/g_{s-r-1,\,r+1}\, g_{n-s-1,\,s+1}$$

Putting

3.73 $$\overline{r,\, n} = g_{n-r-1,\,r+1} \qquad (r \leq n),$$

so that

$$\overline{r,\, r+1} = 1, \qquad \overline{r,\, r+2} = 2 \quad \text{and} \quad \overline{-1,\, n} = g_n,$$

we have simply

3.74 $$({}^{s}_{r}|_{n}) = \overline{r,\, n} \Big/ \overline{r,\, s}\ \overline{s,\, n}.$$

It will be found that 1.42 is satisfied identically. By 1.25,

$$\overline{r,\, r} = 1.$$

We therefore say

$$g_{-1} = 1 \quad (\text{and} \quad g_{-1,\,u} = 1).$$

By 2.64,

3.75 $$g_m = ({}^{0}|_{m})\,({}^{0}|_{m-1,\,1})\,({}^{0}|_{m-2,\,2}) \ldots ({}^{0}|_{1,\,m-1}).$$

Similarly, by 3.72 with $s = m-1$,

$$g_m = ({}^{m-1}|_{m})\, g_{m-1}$$

3.76 $$= ({}^{m-1}|_{m})\,({}^{m-2}|_{m-1})\,({}^{m-3}|_{m-2}) \ldots ({}^{0}|_{1}).$$

By 1.8, reciprocal polytopes have the same g_m. Thus 3.75 and 3.76 are reciprocal formulæ.

It is interesting to note that the first few g's are *rational* functions of the k's (3.45), namely,

3.77 $$\begin{cases} g_{-1} = 1, & \text{implying} \quad g_{-1,\,u} = 1\,; \\ g_0 = 1, & \quad ,, \qquad g_{0,\,u} = 1\,; \\ g_1 = 2, & \quad ,, \qquad g_{1,\,u} = 2\,; \\ g_2 = 2k_1, & \quad ,, \qquad g_{2,\,u} = 2k_{u+1}\,; \\ g_3 = 4\Big/\left(\dfrac{1}{k_1} + \dfrac{1}{k_2} - \tfrac{1}{2}\right), & \quad ,, \qquad g_{3,\,u} = 4\Big/\left(\dfrac{1}{k_{u+1}} + \dfrac{1}{k_{u+2}} - \tfrac{1}{2}\right). \end{cases}$$

The value of g_3 comes by substituting 3.72 in EULER's theorem

$$({}^{0}|_{3}) - ({}^{1}|_{3}) + ({}^{2}|_{3}) = 2.$$

The higher g's are *transcendental* functions.

3.8. In practice, we count the number of vertices of a polytope (given by co-ordinates, as in 3.6), deduce g_m by means of 3.75, and thence $({}^s|_m)$ by 3.72.

For α_m, $({}^0|_m) = m + 1$. So

$$g_m = (m+1)\,!, \qquad g_s = (s+1)\,!, \qquad g_{m-s-1,\, s+1} = (m-s)\,!$$

and

3.81 $$({}^s|_m) = \binom{m+1}{s+1},$$

the elements being α_s.

For β_m, $({}^0|_m) = 2m$, so that $g_m = 2^m m\,!$, $\quad g_{m-s-1,\, s+1} = 2^{m-s-1}(m-s-1)\,!$ and (since, for $s < m$, $\Pi_s = \alpha_s$), $\quad g_s = (s+1)\,!$.
Thus

3.82 $$({}^s|_m) = 2^{s+1}\binom{m}{s+1} \quad (s < m),$$

the elements being again α_s.

γ_m is reciprocal to β_m. So $g_m = 2^m m\,!$ again, and

3.83 $$({}^s|_m) = 2^{m-s}\binom{m}{s} \quad (s > -1),$$

the elements being γ_s.

The elements of δ_m are all γ_s, the number at a vertex being equal to the number of α_{s-1}'s in β_{m-1}, *viz.*, $2^s\binom{m-1}{s}$.

The results for the remaining finite polytopes are as follows :—

	g_m	$({}^0\|_m)$	$({}^1\|_m)$	$({}^2\|_m)$	$({}^3\|_m)$
$\{k\}$	$2k$	k	k		
$\{3, 5\}$	120	12	30	20	
$\{5, 3\}$	120	20	30	12	
$\{3, 3, 5\}$	14400	120	720	1200	600
$\{5, 3, 3\}$	14400	600	1200	720	120
$\{3, 4, 3\}$	1152	24	96	96	24

3.9. The values of the circum-radii of the regular polytopes follow directly from the co-ordinates of the vertices, or can be calculated by means of 2.93. The other radii, ${}_n\mathrm{R}_m$, are then given by 2.73. For all *degenerate* polytopes,

$${}_n\mathrm{R}_m = \infty\,.$$

For the *finite* polytopes, the values are as follows :—

	${}_0\mathrm{R}_m$	${}_1\mathrm{R}_m$	${}_2\mathrm{R}_m$	${}_3\mathrm{R}_m$	${}_n\mathrm{R}_m$
α_m	$\sqrt{\tfrac{1}{2}\left(1-\frac{1}{m+1}\right)}$				$\sqrt{\tfrac{1}{2}\left(\frac{1}{n+1}-\frac{1}{m+1}\right)}$
β_m	$\sqrt{\tfrac{1}{2}}$				$\sqrt{\tfrac{1}{2}\frac{1}{n+1}}$ $[n<m]$
γ_m	$\tfrac{1}{2}\sqrt{m}$				$\tfrac{1}{2}\sqrt{m-n}$ $[n>-1]$
$\{k\}$	$\tfrac{1}{2}\operatorname{cosec}\frac{\pi}{k}$	$\tfrac{1}{2}\cot\frac{\pi}{k}$			
$\{3, 5\}$	$\tfrac{1}{2}5^{\frac{1}{4}}\tau^{\frac{1}{2}}$	$\tfrac{1}{2}\tau$	$\tfrac{1}{2}\sqrt{\tfrac{1}{3}}\tau^2$		$\sqrt{\tfrac{1}{2}\left(\tfrac{1}{2}\tau+\frac{1}{n+1}\right)}$ $[n<3]$
$\{5, 3\}$	$\tfrac{1}{2}\sqrt{3}\tau$	$\tfrac{1}{2}\tau^2$	$\tfrac{1}{2}5^{-\frac{1}{4}}\tau^{\frac{5}{2}}$		$\tau^2\sqrt{\tfrac{1}{2}\left(\tfrac{1}{2}\tau-\frac{1}{\tau^3-n}\right)}$ $[n>-1]$
$\{3, 3, 5\}$	τ	$\tfrac{1}{2}5^{\frac{1}{4}}\tau^{\frac{3}{2}}$	$\sqrt{\tfrac{1}{3}}\tau^2$	$\tfrac{1}{2}\sqrt{\tfrac{1}{2}}\tau^3$	$\sqrt{\tfrac{1}{2}\left(\tau^3+\frac{1}{n+1}\right)}$ $[n<4]$
$\{5, 3, 3\}$	$\sqrt{2}\tau^2$	$\tfrac{1}{2}\sqrt{3}\tau^3$	$5^{-\frac{1}{4}}\tau^{\frac{7}{2}}$	$\tfrac{1}{2}\tau^4$	$\tau^2\sqrt{\tfrac{1}{2}\left(\tau^3-\frac{1}{\tau^3-n}\right)}$ $[n>-1]$
$\{3, 4, 3\}$	1	$\tfrac{1}{2}\sqrt{3}$	$\sqrt{\tfrac{2}{3}}$	$\sqrt{\tfrac{1}{2}}$	

(By 3.61,

$$\tau = \tfrac{1}{2}(\sqrt{5}+1), \qquad \tau^2 = \tfrac{1}{2}(3+\sqrt{5}), \qquad \tau^3 = \sqrt{5}+2,$$
$$\tau^4 = \tfrac{1}{2}(7+3\sqrt{5}), \qquad \tau^5 = \tfrac{1}{2}(5\sqrt{5}+11) \quad \text{and} \quad \tau^7 = \tfrac{1}{2}(13\sqrt{5}+29).)$$

4. *The Generalized Prism.*

4.1. Let

$$(x_1, \ldots x_p), \quad (x_{p+1}, \ldots x_q), \quad (x_{q+1}, \ldots x_r), \quad \text{etc.},$$

be the vertices of certain finite polytopes

4.11 $$\Pi^{(1)}_{m_1}, \quad \Pi^{(2)}_{m_2}, \quad \Pi^{(3)}_{m_3}, \quad \text{etc.},$$

respectively. Then the new polytope whose vertices are

$$(x_1, \ldots x_p\,; \;\; x_{p+1}, \ldots x_q\,; \;\; x_{q+1}, \ldots x_r\,; \;\; \ldots)$$

(in the notation of 3.6) is called the "prism" having the "constituents" 4.11. It is denoted by the symbol

4.12 $$[\Pi^{(1)}_{m_1},\quad \Pi^{(2)}_{m_2},\quad \Pi^{(3)}_{m_3},\quad \ldots].$$

4.2. The following properties are immediate :—

4.21. The prism is uniform if its constituents are uniform and of equal edge-length.

4.22. The order of the constituents is immaterial, and any constituents which are themselves prisms can be replaced by their own constituents.

4.23. Constituents α_0 can be omitted.

4.24. A number n of constituents α_1 ($= \gamma_1$) can be replaced by γ_n.

4.25. Constituents γ_n, $\gamma_{n'}$, $\gamma_{n''}$, ... can be replaced by $\gamma_{n+n'+n''+\ldots}$.

4.26. A prism with only one constituent is that constituent itself.

4.27. The number of dimensions of the prism is the sum of the numbers of dimensions of the constituents.

4.28. The number of vertices is the product of the numbers of vertices of the constituents.

4.29. The square of the circum-radius is equal to the sum of the squares of the circum-radii of the constituents.

In symbols, 4.27, 4.28, 4.29 can be written :

4.27 $$m = m_1 + m_2 + m_3 + \ldots,$$

4.28 $$({}^0|_m) = ({}^0|^{(1)}_{m_1})\,({}^0|^{(2)}_{m_2})\,({}^0|^{(3)}_{m_3})\ \ldots,$$

4.29 $$({}_0\mathrm{R}_m)^2 = ({}_0\mathrm{R}^{(1)}_{m_1})^2 + ({}_0\mathrm{R}^{(2)}_{m_2})^2 + ({}_0\mathrm{R}^{(3)}_{m_3})^2 + \ldots$$

4.3. It is also true that the m-dimensional content of the prism is equal to the product of the contents of the constituents ; and that the magnitude of the vertex angle, measured as a fraction of the total angle at a point in m dimensions, is equal to the product of the magnitudes of the vertex angles of the constituents.

These two theorems are respectively very easy and very hard to prove. Neither is required later, so the proofs are omitted.

4.4. As three-dimensional examples of the generalized prism (a, b, c, h being lengths) :

4.41 $$[\alpha_1 a,\quad \alpha_1 b,\quad \alpha_1 c]$$

is the rectangular solid of edges a, b, c ; and

$$[\{k\},\quad \alpha_1 h]$$

is the right prism of height h on a regular k-gon (of side 1) as base. This right prism is uniform ("pure Archimedean") if $h = 1$.

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The four-dimensional uniform prism

$$[\{k\}, \quad \{k'\}]$$

is bounded by k $[\{k'\}, \alpha_1]$'s and k' $[\{k\}, \alpha_1]$'s. It can be constructed as follows. Take the k k'-gonal prisms $[\{k'\}, \alpha_1]$ and place them base to base, bending them about the planes of the intermediate bases until the two extreme bases meet, the whole forming a kind of ring. Make an analogous ring by means of the k' k-gonal prisms $[\{k\}, \alpha_1]$. Each of these rings has kk' untouched squares, and the complete polytope is made by interlocking the rings in such a way that the two sets of squares are brought into coincidence.

4.5. If a generalized prism is uniform (4.21), its vertex figure is obtained by taking the vertex figures of the constituents, in independent spaces, and joining every vertex of every one (of these vertex figures) to every vertex of every other, by lines of length $\sqrt{2}$ (*i.e.*, by β_1's); this construction being possible in $m_1+m_2+m_3+\ldots-1$ dimensions.

If the uniform prism has only two constituents, $\Pi^{(1)}_{m_1}$ and $\Pi^{(2)}_{m_2}$, we give its vertex figure the special symbol

$$(\Pi_{m_1-1,\,1}^{(1)} \overset{}{\underset{\sqrt{2}}{\text{———}}} \Pi_{m_2-1,\,1}^{(2)}).$$

e.g.,

$$(\alpha_1 \underset{\sqrt{2}}{\text{———}} \alpha_0)$$

denotes the isosceles triangle, of sides 1, $\sqrt{2}$, $\sqrt{2}$, which is the vertex figure of the triangular prism

$$[\alpha_2, \ \alpha_1].$$

4.6. The elements of the prism 4.12 consist of all possible prisms of the form

4.61 $$[\Pi^{(1)}_{r_1}, \quad \Pi^{(2)}_{r_2}, \quad \Pi^{(3)}_{r_3}, \ldots],$$

where $\Pi^{(1)}_{r_1}$ is an element of $\Pi^{(1)}_{m_1}$, and so on. By considering the number of ways in which the element 4.61 can occur, we find

4.62 $$({}^{r}|_{m}) = \sum_{r_1=0}^{m_1}\sum_{r_2=0}^{m_2}\sum_{r_3=0}^{m_3}\ldots({}^{r_1}|{}^{(1)}_{m_1})\,({}^{r_2}|{}^{(2)}_{m_2})\,({}^{r_3}|{}^{(3)}_{m_3})\ldots \quad (r_1+r_2+r_3+\ldots=r).$$

(In verification of 1.22, we have

$$\sum_{r=0}^{m}(-1)^r({}^{r}|_{m}) = \sum_{r_1=0}^{m_1}\sum_{r_2=0}^{m_2}\sum_{r_3=0}^{m_3}\ldots(-1)^{r_1+r_2+r_3+\ldots}({}^{r_1}|{}^{(1)}_{m_1})\,({}^{r_2}|{}^{(2)}_{m_2})\,({}^{r_3}|{}^{(3)}_{m_3})\ldots$$

$$= \sum_{r_1=0}^{m_1}(-1)^{r_1}({}^{r_1}|{}^{(1)}_{m_1})\,.\sum_{r_2=0}^{m_2}(-1)^{r_2}({}^{r_2}|{}^{(2)}_{m_2})\,.\sum_{r_3=0}^{m_3}(-1)^{r_3}({}^{r_3}|{}^{(3)}_{m_3})\ldots$$

$$= 1\,.\,1\,.\,1\ldots = 1.)$$

In particular, the number of bounding figures is

4.63 $$({}^{m-1}|_{m}) = ({}^{m_1-1}|{}^{(1)}_{m_1}) + ({}^{m_2-1}|{}^{(2)}_{m_2}) + ({}^{m_3-1}|{}^{(3)}_{m_3}) + \ldots;$$

the bounding figures being

4.64 $[\Pi^{(1)}_{m_1-1}, \Pi^{(2)}_{m_2}, \Pi^{(3)}_{m_3}, \ldots], \quad [\Pi^{(1)}_{m_1}, \Pi^{(2)}_{m_2-1}, \Pi^{(3)}_{m_3}, \ldots], [\Pi^{(1)}_{m_1}, \Pi^{(2)}_{m_2}, \Pi^{(3)}_{m_3-1}, \ldots], \ldots.$

4.7. Suppose the prism 4.12 to have been reduced (if necessary) in accordance with **4.24** and 4.25, so that not more than one constituent is a γ. Let $g^{(1)}_{m_1}$, $g^{(2)}_{m_2}$, $g^{(3)}_{m_3}$, etc., be the orders of the groups of symmetries of the constituents. Then the order of the group of symmetries of the prism is evidently

4.71 $$g_m = \lambda g^{(1)}_{m_1} g^{(2)}_{m_2} g^{(3)}_{m_3} \ldots ;$$

where $\lambda = 1$ if the constituents are all different, but $\lambda = N\,!\; N'\,! \ldots$ if N constituents are identical, N′ others identical, and so on, since the identical constituents can be permuted among themselves.

For instance, if

$$\Pi_{p+q} = [\alpha_p, \alpha_q] \qquad (p + q > 0^*),$$

4.72 $$g_{p+q} = (1 + \varepsilon_{pq})\,(p+1)\,!\;(q+1)\,!$$

where

4.73 $$\varepsilon_{pq} = \begin{cases} 0 & \text{if} \quad p \neq q, \\ 1 & \text{if} \quad p = q. \end{cases}$$

4.8. Precisely as in **4.1**, we can define the " degenerate prism "

4.81. $$\Pi_{m+1} = [\Pi^{(1)}_{m_1+1}, \quad \Pi^{(2)}_{m_2+1}, \quad \Pi^{(3)}_{m_3+1}, \ldots]$$

whose constituents

$$\Pi^{(1)}_{m_1+1}, \quad \Pi^{(2)}_{m_2+1}, \quad \Pi^{(3)}_{m_3+1}, \text{ etc.,}$$

are *degenerate* polytopes.

4.21, 4.22 and 4.26 still apply ; but 4.24, 4.25 and 4.27 must be replaced respectively by :—

4.82. A number n of constituents δ_2 can be replaced by δ_{n+1}.

4.83. Constituents δ_{n+1}, $\delta_{n'+1}$, $\delta_{n''+1}$, … can be replaced by $\delta_{n+n'+n''+\ldots+1}$.

4.84. The number of dimensions of the space filled by the prism is the sum of the numbers of dimensions of the spaces filled by the constituents.

The description 4.5 of the vertex figure of a uniform prism still applies, except that the vertex figures of the constituents now lie in mutually perpendicular spaces (of m_1, m_2, m_3, … dimensions) *having a common point*, which point is the centre of each constituent's vertex figure ; this construction being possible in $m = m_1 + m_2 + m_3 + \ldots$ dimensions.

4.9. The elements of the degenerate prism 4.81 consist of all possible (finite) prisms of the form 4.61, where now $\Pi^{(p)}_{r_p}$ is an element of $\Pi^{(p)}_{m_p+1}$ $(r_p \leq m_p)$. In particular, the bounding figures are of the form 4.12.

* In order to cover the case $p = 0 = q$, 4.72 must be replaced by

$$g_{p+q} = (1 + \varepsilon_{pq} - \varepsilon_{p0}\varepsilon_{q0})\,(p+1)\,!\;(q+1)\,!$$

To take a simple example,

$$[\delta_2 a, \quad \delta_2 b, \quad \delta_2 c]$$

is the partition of three-dimensional space into rectangular solids 4.41. In particular (by 4.82 and 4.83)

$$[\delta_2, \quad \delta_2, \quad \delta_2] = [\delta_3, \quad \delta_2] = \delta_4.$$

(" Prisms " whose constituents are partly finite and partly degenerate may be called " semi-degenerate," but are uninteresting.)

5. *Simple Truncation.*

5.1. A polytope which consists of the portion of m-space common to two concentric and actually reciprocal regular polytopes (Π_m and Π'_m) is called a " truncation " (of Π_m or Π'_m). If the radius of reciprocation has the particular value ${}_nR_m$, so that the reciprocating sphere-analogue touches the n-dimensional elements of Π_m and (therefore) the $(m-n-1)$-dimensional elements of Π'_m, the truncation is said to be " simple," and is denoted by

5.11 $$t_n \Pi_m \quad \text{or} \quad t_{m-n-1} \Pi'_m.$$

$t_n \Pi_m$ could have been defined simply as the polytope whose vertices are the centres of the Π_n's of Π_m. But the mental picture of a fixed Π_m and a gradually diminishing reciprocal Π'_m is useful.

Genuine truncations are obtained for values of n from 0 to $m-1$.

5.12 $$t_0 \Pi_m = \Pi_m \quad \text{and} \quad t_{m-1} \Pi_m = \Pi'_m.$$

$t_m \Pi_m$ is merely a point, namely, the centre of Π_m. By 5.11, $t_m \Pi_m$ is the same as $t_{-1} \Pi'_m$; so we must take

$$t_{-1} \Pi_m$$

to mean the centre too.

As a familiar example of a truncation, $t_1 \beta_3$ (or $t_1 \gamma_3$) is the cuboctahedron. Still more simply

5.13 $$t_1 \{k\} = \{k\}.$$

5.2. The properties of $t_n \Pi_m$ are functions of the properties of Π_m, and will be distinguished from them by the suffix n, *e.g.*, $({}^1|_m)_n$ means the number of edges of $t_n \Pi_m$.

It follows from the definition (5.1) that

5.21 $$({}^0|_m)_n = ({}^n|_m).$$

Consider a fixed Π_m and a gradually shrinking reciprocal Π'_m (obtained by means of a gradually shrinking reciprocating sphere-analogue). While the radius of reciprocation is diminishing from the value ${}_0R_m$, Π_m has all its corners cut off and replaced by new bounding figures similar to $\Pi_{m-1,1}$. These new bounding figures increase in size until the position corresponding to $t_1 \Pi_m$ is reached. Then they too begin to be truncated, appearing as $t_1 \Pi_{m-1,1}$'s in $t_2 \Pi_m$. Thus it is clear that the bounding figures of $t_n \Pi_m$ are of two kinds,

5.22 $$t_n \Pi_{m-1} \quad \text{and} \quad t_{n-1} \Pi_{m-1,1},$$

corresponding respectively to the bounding figures and vertices of Π_m.

The $(m-2)$-dimensional elements of $t_n \Pi_m$, being the bounding figures of its bounding figures, must consequently be of the three kinds

$$t_n \Pi_{m-2}, \quad t_{n-1} \Pi_{m-2,1}, \quad t_{n-2} \Pi_{m-2,2}.$$

Similarly, or by induction, it is easy to see that all possible s-dimensional elements are of the form

5.23 $$t_{n-u} \Pi_{s,u},$$

for a certain set of values of u.

In order that $\Pi_{s,u}$ may have a meaning,

5.24 $$0 \leq u \leq m - s;$$

and in order that $t_{n-u} \Pi_{s,u}$ may be a genuine truncation,

$$0 \leq n - u \leq s - 1,$$

i.e.,

5.25 $$n - s + 1 \leq u \leq n.$$

The number of elements $t_{n-u} \Pi_{s,u}$, for each u, is equal to the number of ways in which the figure $\Pi_{s,u}$ can occur in Π_m. Now, $\Pi_{s,u}$ is an s-dimensional element of $\Pi_{m-u,u}$, which, being the uth vertex figure of Π_m, indicates the form of the neighbourhood of an element Π_{u-1} (2.6). Hence $\Pi_{m-u,u}$ occurs $({}^{u-1}|_m)$ times, and so $\Pi_{s,u}$ must occur $({}^{u-1}|_m)({}^{s}|_{m-u,u})$ times.

Thus the total number of s-dimensional elements of $t_n \Pi_m$ is

5.26 $$({}^{s}|_m)_n = \sum_{0,\, n-s+1}^{m-s,\, n} ({}^{u-1}|_m)({}^{s}|_{m-u,u}) \qquad (s > 0)$$

where $\sum\limits_{0,\, n-s+1}^{m-s,\, n}$ stands for $\sum\limits_{u=\max.(0,\, n-s+1)}^{u=\min.(m-s,\, n)}$,

the typical element for each u being $t_{n-u} \Pi_{s,u}$.

Note that 5.26 does not include 5.21.

2.61 and 2.63 respectively enable 5.26 to be exhibited in two alternative forms:

5.27 $$({}^{s}|_{m})_n = \sum_{0,\, n-s+1}^{m-s,\, n} ({}^{u-1}|_{m})\,({}^{s+u}_{u-1}|_{m}),$$

5.28 $$({}^{s}|_{m})_n = \sum_{0,\, n-s+1}^{m-s,\, n} ({}^{u-1}|_{s+u})\,({}^{s+u}|_{m}).$$

By 5.21, applied to $\sigma \equiv t_{n-u}\,\Pi_{s,\,u}$; for each u, $({}^{0}|{}^{\sigma}_{s})_n = ({}^{n-u}|_{s,\,u})$.
Hence, by 1.41 with $r = 0$,

$$\begin{aligned}({}^{0}|_{m})_n\,({}^{s}_{0}|_{m})_n &= \sum_{\sigma} ({}^{0}|{}^{\sigma}_{s})_n\,({}^{s}|{}^{\sigma}_{m})_n \\ &= \sum_{0,\, n-s+1}^{m-s,\, n} ({}^{n-u}|_{s,\,u})\,({}^{u-1}|_{m})\,({}^{s}|_{m-u,\,u}) && (5.26) \\ &= \sum_{0,\, n-s+1}^{m-s,\, n} ({}^{u-1}|_{m})\,({}^{n-u}|_{m-u,\,u})\,({}^{s}_{n-u}|_{m-u,\,u}) && (1.41) \\ &= \sum_{0,\, n-s+1}^{m-s,\, n} ({}^{u-1}|_{n})\,({}^{n}|_{m})\,({}^{s}_{n-u}|_{m-u,\,u}) && (2.63) \\ &= ({}^{n}|_{m}) \sum_{0,\, n-s+1}^{m-s,\, n} ({}^{u-1}|_{n})\,({}^{s+u}_{n}|_{m}). && (2.62)\end{aligned}$$

Finally, using 5.21,

$$({}^{s}_{0}|_{m})_n = \sum_{0,\, n-s+1}^{m-s,\, n} ({}^{u-1}|_{n})\,({}^{s+u}_{n}|_{m}),$$

i.e.,

5.29 $$({}^{s-1}|_{m-1,\,1})_n = \sum_{0,\, n-s+1}^{m-s,\, n} ({}^{u-1}|_{n})\,({}^{s+u-n-1}|_{m-n-1,\,n+1}).$$

5.3. Since Π_m and Π'_m have the same symmetries, these symmetries must belong also to $t_n\,\Pi_m$. The equivalence of the vertices of $t_n\,\Pi_m$ therefore follows from the equivalence of the Π_n's of Π_m (3.2). Since simple truncations are bounded by simple truncations, it is thus obvious (by induction) that $t_n\,\Pi_m$ is uniform.

We now seek to justify the assumption that the vertex figure of $t_n\,\Pi_m$ is

5.31 $$[\Pi'_n, \quad \Pi_{m-n-1,\,n+1}],$$

Π'_n having the special meaning assigned in 3.32.

This is trivially true when $n = 0$ or $n = m-1$, and therefore when $m = 2$, so we have a basis for induction. Accordingly, we assume $[\Pi'_n,\ \Pi_{s-n-1,\,n+1}]$ to be the vertex figure of $t_n\,\Pi_s$ for $0 \leq n < s < m$, and consequently

5.32 $$[\Pi'_{n-u,\,u}, \quad \Pi_{s+u-n-1,\,n+1}]$$

to be the vertex figure of $t_{n-u}\,\Pi_{s,\,u}$ for $0 \leq n-u < s < m$.

For each value of u satisfying 5.24 and 5.25, the vertex figure of $t_n \Pi_m$ possesses, by 5.29,

$$({}^{u-1}|_n)\ ({}^{s+u-n-1}|_{m-n-1,\, n+1})$$

$(s-1)$-dimensional elements 5.32. We have to show that these are precisely the $(s-1)$-dimensional elements of the prism 5.31.

By 4.61, the typical element of 5.31 is

5.33 $$[\Pi'_{r_1,\, n-r_1}, \qquad \Pi_{r_2,\, n+1}],$$

$\Pi'_{r_1,\, n-r_1}$ being (by 3.31, with n for m and $n-r_1$ for u) the r_1-dimensional element of Π'_n. Since the number of $\Pi'_{r_1,\, n-r_1}$'s in Π'_n is $({}^{n-r_1-1}|_n)$,* while the number of $\Pi_{r_2,\, n+1}$'s in $\Pi_{m-n-1,\, n+1}$ is $({}^{r_2}|_{m-n-1,\, n+1})$, it follows (by 4.62) that the number of (r_1+r_2)-dimensional elements 5.33 in 5.31 is

$$({}^{n-r_1-1}|_n)\ ({}^{r_2}|_{m-n-1,\, n+1}),$$

where

5.34 $$\begin{cases} 0 \leq r_1 \leq n, \\ 0 \leq r_2 \leq m-n-1. \end{cases}$$

We can identify the elements 5.32 and 5.33, and the number of times they occur, by putting

$$\begin{cases} r_1 = n-u, \\ r_2 = s+u-n-1. \end{cases}$$

The inequalities 5.34 then become

$$\begin{cases} 0 \leq u \leq n, \\ n-s+1 \leq u \leq m-s, \end{cases}$$

which are together equivalent to 5.24 and 5.25.

The argument, that 5.31 is consequently the vertex figure of $t_n\Pi_m$, (if not entirely justifiable, as assuming that the elements of such a prism cannot be re-arranged to form a new polytope,) appears convincing, especially as the vertex figure of $t_n \Pi_m$ must possess the symmetries of both Π_n and $\Pi_{m-n-1,\, n+1}$.

Putting this result in terms of Schläfli symbols, the vertex figure of

$$t_n \{k_1, k_2, \ldots k_{m-2}, k_{m-1}\}$$

is

5.35 $$\left[\ \{k_{n-1}, k_{n-2}, \ldots k_2, k_1\}\, 2 \cos\frac{\pi}{k_n}, \qquad \{k_{n+2}, k_{n+3}, \ldots k_{m-2}, k_{m-1}\}\, 2\cos\frac{\pi}{k_{n+1}}\right].$$

(The two constituents of this prism are the vertex figures of

$$\{k_n, k_{n-1}, \ldots k_2, k_1\} \quad\text{and}\quad \{k_{n+1}, k_{n+2}, \ldots k_{m-2}, k_{m-1}\}$$

respectively.)

* By 1.351 (with n for m, r_1 for s' and consequently $n-r_1-1$ for s), the number of $(n-r_1-1)$-dimensional elements of Π_n is the same as the number of r_1-dimensional elements of Π'_n

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5.4. Let O be the centre of a Π_{n+1} of Π_m; Q the centre of a Π_{n-1} belonging to this Π_{n+1}; and P, P′ the centres of the two Π_n's of the Π_{n+1} which meet at this Π_{n-1}. Let

$$(a)_n$$

be the edge-length of the actual truncation $t_n \Pi_m$. Then we have $PP' = (a)_n$, $OP = {}_n R_{n+1}$, $OQ = {}_{n-1}R_{n+1}$, $PQ = {}_{n-1}R_n$, and OPQ is a right angle.

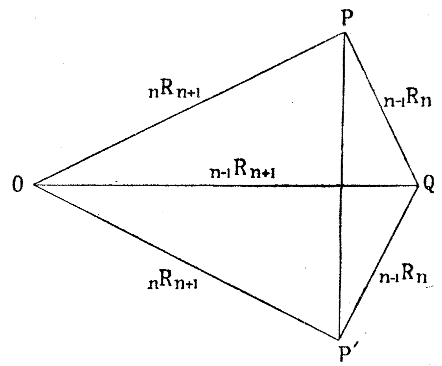

FIG. 2.

Hence

5.41 $$(a)_n = 2\,{}_{n-1}R_n\,{}_n R_{n+1}/{}_{n-1}R_{n+1}.$$

We shall in future regard $t_n \Pi_m$ as having been magnified until its edge-length is unity.

Let $({}_0R_m)_n$ be the circum-radius of $t_n \Pi_m$ (for unit edge). Then, since obviously

$$({}_0R_m)_n\,(a)_n = {}_n R_m,$$

5.41 gives

5.42 $$({}_0R_m)_n = {}_{n-1}R_{n+1}\,{}_n R_m/2\,{}_{n-1}R_n\,{}_n R_{n+1}.$$

5.5. The following simple truncations happen to be regular, as may be seen (by 4.25) from their vertex figures, here placed alongside :—

$t_1\alpha_3 = \beta_3$	$[\alpha_1, \alpha_1] = \beta_2$
$t_1\delta_3 = \delta_3$	$[\beta_1, \beta_1] = \beta_2\sqrt{2}$
$t_1\beta_4 = t_2\gamma_4 = \{3, 4, 3\}$	$[\alpha_1, \beta_2] = [\beta_2, \alpha_1] = \gamma_3$
$t_1\{3, 3, 4, 3\} = t_2\delta_5 = t_3\,\{3, 4, 3, 3\} = \{3, 4, 3, 3\}$	$[\alpha_1, \gamma_3] = [\beta_2, \beta_2] = [\gamma_3, \alpha_1] = \gamma_4$

By 4.71, the order of the group of symmetries of the vertex figure 5.31 is

$$(g_{m-1,\,1})_n = \lambda\, g_n\, g_{m-n-1,\; n+1},$$

where

$$\lambda = 1 \quad \text{in general,}$$

but

$$\lambda = 2 \quad \text{if} \quad \Pi'_n = \Pi_{m-n-1,\, n+1}$$

(which implies

$$m = 2n + 1 \quad \text{and} \quad \Pi'_m = \Pi_m).$$

Also

$$\lambda = \binom{m-1}{n} \quad \text{if} \quad \Pi'_n = \gamma_n\, a \quad \text{and} \quad \Pi_{m-n-1,\, n+1} = \gamma_{m-n-1}\, a$$

(which implies

$$\Pi_m = \{3, \ldots 3, 4, k_n, k_{n+1}, 4, 3, \ldots 3\} \quad \text{with} \quad k_n = k_{n+1}),$$

since this case was excluded in formulating 4.71. Hence, by **2.41**, **5.21** and **3.72**, the order of the group of symmetries of $t_n \Pi_m$ is

5.51 $$(g_m)_n = \lambda g_m,$$

where

5.52 $$\begin{cases} \lambda = 1 & \text{in general,} \\ \lambda = 2 & \text{for} \quad t_n \alpha_{2n+1}, \\ \lambda = 3 & \text{for} \quad t_1\beta_4 = t_2\gamma_4 \ (= \{3, 4, 3\}). \end{cases}$$

5.6. Here is a summary of the chief properties of the simple truncations (excluding those truncations which are regular) :—

Π_m	$\Pi_{m-1,\,1}$	$({}^0\vert{}_m)$	g_m	${}_0\mathrm{R}_m$
$t_n\alpha_m$	$[\alpha_n, \alpha_{m-n-1}]$	$\binom{m+1}{n+1}$	$(1 + \varepsilon_{m\,(2n+1)})\,(m+1)\,!$	$\sqrt{\dfrac{(m-n)(n+1)}{2(m+1)}}$
$t_n\beta_m$	$[\alpha_n, \beta_{m-n-1}]$	$2^{n+1}\binom{m}{n+1}$	$2^m\,m\,!$	$\sqrt{\dfrac{n+1}{2}}$ $(n<m-1)$
$t_n\gamma_m$	$[\beta_n, \alpha_{m-n-1}]$	$2^{m-n}\binom{m}{n}$	$2^m\,m\,!$	$\sqrt{\dfrac{m-n}{2}}$ $(n>0)$
$t_n\delta_m$	$[\beta_n, \beta_{m-n-1}]$	∞	∞	∞
$t_1\{3, 5\} = t_1\{5, 3\}$	$[\alpha_1\tau, \alpha_1]$	30	120	τ
$t_1\{3, 6\} = t_1\{6, 3\}$	$[\alpha_1\sqrt{3}, \alpha_1]$	∞	∞	∞
$t_1\{3, 3, 5\} = t_2\{5, 3, 3\}$	$[\{5\}, \alpha_1]$	720	14400	$5^{\frac{1}{4}}\tau^{\frac{3}{2}}$
$t_1\{5, 3, 3\} = t_2\{3, 3, 5\}$	$[\alpha_2, \alpha_1\tau]$	1200	14400	$\sqrt{3}\tau^2$
$t_1\{3, 4, 3\} = t_2\{3, 4, 3\}$	$[\alpha_2\sqrt{2}, \alpha_1]$	96	1152	$\sqrt{3}$
$t_1\{3, 4, 3, 3\} = t_3\{3, 3, 4, 3\}$	$[\alpha_3\sqrt{2}, \alpha_1]$	∞	∞	∞
$t_2\{3, 3, 4, 3\} = t_2\{3, 4, 3, 3\}$	$[\alpha_2\sqrt{2}, \alpha_2]$	∞	∞	∞

5.7. Let

$$(1^p, 0^q)$$

stand for

$$(1, \ldots 1, 0, \ldots 0) \quad \text{with} \quad p \text{ ones} \quad \text{and} \quad q \text{ zeros.}$$

If $p > 0$ and $q > 0$, the $\binom{p+q}{p}$ points

$$(1^p, 0^q)$$

are the vertices of $t_{p-1}\alpha_{p+q-1}\sqrt{2}$ $(= t_{q-1}\alpha_{p+q-1}\sqrt{2})$. For, of these points, those nearest to $(1^p;\ 0^q)$ are $(1^{p-1}, 0;\ 1, 0^{q-1})$, namely the vertices of $[\alpha_{p-1}, \alpha_{q-1}]\sqrt{2}$.

Again, the $2^p\binom{p+q}{p}$ points

$$\pm (1^p,\ 0^q)$$

are the vertices of $t_{p-1}\beta_{p+q}\sqrt{2}$ $(= t_q\gamma_{p+q}\sqrt{2})$. For, of these points, those nearest to $(1^p;\ 0^q)$ are $(1^{p-1},\ 0;\ \pm 1,\ 0^{q-1})$, namely the vertices of $[\alpha_{p-1},\ \beta_q]\sqrt{2}$.

Thus, and similarly, we have the following co-ordinates :—

$t_n\alpha_m\sqrt{2}$: $(1^{n+1},\ 0^{m-n})$ or $(1^{m-n},\ 0^{n+1})$.

$t_n\beta_m\sqrt{2}$: $\pm (1^{n+1},\ 0^{m-n-1})$ $(n < m-1)$.

$t_n\gamma_m\sqrt{2}$: $\pm (1^{m-n},\ 0^n)$ $(n > 0)$.

$t_n\delta_m\sqrt{2}$: $(1^n,\ 0^{m-n-1})\ (\text{mod. } 2)$ or $(1^{m-n-1},\ 0^n)\ (\text{mod. } 2)$ $\quad (0<n<m-1)$.

$t_1\{3,\ 5\}2\tau^{-1}$* : $\left\{\begin{array}{l}\pm(\tau,\ 1,\ \tau^{-1})',\\ \pm(2,\ 0,\ 0)\end{array}\right.$

or $\left\{\begin{array}{l}(\tau,\ 0,\ -\tau^{-1},\ -1)',\\ (1,\ 1,\ -1,\ -1),\\ (1,\ \tau^{-1},\ 0,\ -\tau)'.\end{array}\right.$

$t_1\{3,\ 6\}\sqrt{2}$: $(1,\ 0,\ -1)\ (\text{mod. } 2);\ x_1+x_2+x_3=0$

or $(1,\ 0,\ 0)\ (\text{mod. } 2);\ x_1+x_2+x_3=1$.

$t_1\{3,\ 3,\ 5\}2\tau^{-1}$† :

$\pm(2\tau,\ 2,\ 0,\ 0)$	(48 points),
$\pm(2\tau,\ \tau,\ 1,\ \tau^{-1})'$	(192 ,,),
$\pm(\tau^2,\ 2,\ \tau,\ 1)'$	(192 ,,),
$\pm(\tau^2,\ \sqrt{5},\ \tau,\ 0)'$	(96 ,,),
$\pm(\sqrt{5}\tau,\ 1,\ \tau^{-1},\ 0)'$	(96 ,,),
$\pm(\tau^2,\ \tau^2,\ \tau^{-1},\ \tau^{-1})$	(96 ,,).
	(720).

* *Cf.* Schoute's "Analytical treatment of the polytopes ..." (*loc. cit.* in Preface), § 123.

† *Ibid.*, § 160.

$$t_1\{5,3,3\}\,2\tau^{-2}{*}:\quad\begin{cases}\pm(2\tau,\,2\tau^{-1},\,0,\,0) & (48\ \text{points}),\\ \pm(2,\,2,\,2,\,0) & (32\ \ ,,\ \ ),\\ \pm(3,\,1,\,1,\,1) & (64\ \ ,,\ \ ),\\ \pm(\sqrt{5},\,\sqrt{5},\,1,\,1) & (96\ \ ,,\ \ ),\\ \pm(3,\,\tau,\,\tau^{-1},\,0)' & (96\ \ ,,\ \ ),\\ \pm(\tau^2,\,\sqrt{5},\,\tau^{-2},\,0)' & (96\ \ ,,\ \ ),\\ \pm(\tau^2,\,2,\,1,\,\tau^{-2})' & (192\ \ ,,\ \ ),\\ \pm(\sqrt{5},\,2,\,\tau,\,\tau^{-1})' & (192\ \ ,,\ \ ),\\ \pm(2\tau,\,1,\,\tau^{-1},\,\tau^{-2})' & (192\ \ ,,\ \ ),\\ \pm(\tau^2,\,\tau,\,2\tau^{-1},\,1)' & (192\ \ ,,\ \ ),\end{cases}$$

$$(1200).$$

$$t_1\{3,4,3\}\,\sqrt{2}\dagger\ :\quad \pm(2,\,1,\,1,\,0).$$

$$t_1\{3,4,3,3\}\,\sqrt{2}:\quad (2,\,\pm1,\,\pm1,\,0)\ (\text{mod. }4).$$

$$t_2\{3,3,4,3\}\,2\ :\quad\begin{cases}(\pm2,\,\pm2,\,\pm2,\,0)\ (\text{mod. }6),\\ (\pm3,\,\pm1,\,\pm1,\,\pm1)\ (\text{mod. }6).\end{cases}$$

5.8. The general theory of truncation can be extended to the case where Π_m, though not regular, has a uth vertex figure, u being greater than n. For in this case, by 3.23, the Π_n's are still equivalent, besides being regular and equal. We now define

$$t_n\,\Pi_m$$

as having for vertices the centres of these Π_n's. Just as in 5.3, we can show that $t_n\,\Pi_m$ is uniform, its vertex figure being

$$[\Pi'_n,\ \Pi_{m-n-1,\,n+1}].$$

Formulæ 5.42 and 5.21 continue to apply; and so does 5.28, provided we allow $t_{n-u}\,\Pi_{s,\,u}$ to take several forms (for the same value of u) corresponding to the various kinds of Π_{s+u} which may occur in Π_m.

5.9. By 4.21 and 5.35, $t_1 t_n\,\{k_1, k_2, \ldots, k_{m-2}, k_{m-1}\}$ exists if $k_n = k_{n+1}$, and then its vertex figure is found to be

$$\left[\left(\{k_{n-2}, k_{n-3}, \ldots, k_2, k_1\}\,2\cos\frac{\pi}{k_{n-1}}\ \underset{\sqrt{2}}{\text{———}}\ \{k_{n+3}, k_{n+4}, \ldots, k_{m-2}, k_{m-1}\}\,2\cos\frac{\pi}{k_{n+2}}\right),\ \alpha_1\,2\cos\frac{\pi}{k_n}\right],$$

* *Ibid.*, § 160.

† *Ibid.*, § 144.

in the notation of 4.5.* Except in the trivial case of

$$t_1 t_1 \delta_3 = t_1 \delta_3 = \delta_3,$$

it always happens that $k_n = 3$. The actual cases are tabulated below:

Π_m.	$\Pi_{m-1,\,1}$.	$({}^0\vert_m)$.	${}_0R_m$.†
$t_1 t_n \alpha_m$	$[(\alpha_{n-1} \overline{\sqrt{2}}\, \alpha_{m-n-2}), \alpha_1]$	$\binom{m+1}{2}\binom{m-1}{n}$	$\sqrt{\left(1+2n-\frac{2(n+1)^2}{m+1}\right)}$
$t_1 t_n \beta_m \quad (n < m-2)$	$[(\alpha_{n-1} \overline{\sqrt{2}}\, \beta_{m-n-2}), \alpha_1]$	$2^{n+2}\binom{m}{2}\binom{m-2}{n}$	$\sqrt{(2n+1)}$
$t_1 t_n \gamma_m \quad (n > 1)$	$[(\beta_{n-1} \overline{\sqrt{2}}\, \alpha_{m-n-2}), \alpha_1]$	$2^{m-n+1}\binom{m}{2}\binom{m-2}{n-1}$	$\sqrt{(2m-2n-1)}$
$t_1 t_n \delta_m \quad (1 < n < m-2)$	$[(\beta_{n-1} \overline{\sqrt{2}}\, \beta_{m-n-2}), \alpha_1]$	∞	∞
$t_1 t_1 \{3, 3, 5\}$	$[(\alpha_1 \tau \overline{\sqrt{2}}\, \alpha_0), \alpha_1]$	3600	$\sqrt{(19+8\sqrt{5})}$

6. *$h\gamma_m$ and $h\delta_m$, $\alpha_m h$ and $e\alpha_m$.*

6.1. In order to establish the uniformity of a polytope whose vertices are a given set of points in m dimensions, we have to prove :—

(i) That the points are equivalent.

(ii) That those points which are nearest to a particular point A (of the set) are sufficient to determine an $(m-1)$-dimensional polytope.

(iii) That the vertices of a typical bounding figure of each kind (*i.e.*, one typical bounding figure from every set which are known to be equivalent among themselves) of this $(m-1)$-dimensional polytope, along with the point A and certain other points of the original set, are the complete set of vertices of some uniform $(m-1)$-dimensional polytope.

This practical rule will be applied to special cases in the present chapter, and in chapter 9.

6.2. It is well known that the vertices of the cube (γ_3) are also the vertices of two concentric tetrahedra $(\alpha_3\sqrt{2})$. It is almost equally obvious that the vertices of γ_4 are also the vertices of two concentric $\beta_4\sqrt{2}$'s. We accordingly write

$$h\gamma_4 = \beta_4,$$

$$h\gamma_3 = \alpha_3,$$

and seek a generalization, $h\gamma_m$ (short for "hemi-γ_m").

* Thus $t_1 t_n \Pi_m$ exists if $\Pi_{1,\,n} = \Pi_{1,\,n+1}$, and then its vertex figure is $[(\Pi'_{n-1} \overline{\sqrt{2}}\, \Pi_{m-n-2,\,n+2}), \Pi_{1,\,n}]$.

† Since $k_n = 3$, this is simply $\sqrt{4\,({}_0R_m)_n^2 - 1}$, $({}_0R_m)_n$ being given in the last column of 5.6.

In considering groups of symmetries, we have to suppose $h\gamma_2 = \{2\}$ (the "digon"). For all other purposes, we drop a dimension and say

$$h\gamma_2 = \alpha_1.$$

Similarly

$$h\gamma_1 = \alpha_0.$$

In the notation of 5.7, the $\sum_0^m \binom{m}{r} = 2^m$ points

6.21 $$(1^r, 0^{m-r}); \qquad r = 0, 1, 2, \dots m,$$

are the vertices of γ_m, since they can be obtained from $\pm(1^m)$ by adding 1 to every co-ordinate and then halving throughout.

We define

$$h\gamma_m \sqrt{2}$$

as having for vertices half these points, namely the 2^{m-1} points

6.22 $$(1^{2r}, 0^{m-2r}); \qquad r = 0, 1, \dots \left[\frac{m}{2}\right],$$

where, as usual, $\left[\frac{m}{2}\right]$ means "the greatest integer not greater than $\frac{m}{2}$."

Applying the rule 6.1, we have :—

(i) These 2^{m-1} points are equivalent, since the operation of subtracting two of the co-ordinates from unity, while leaving the set of points unchanged as a whole, changes, after a sufficient number of applications, any point of the set into any other.

(ii) Taking the typical point A to be $(1^0, 0^m)$ or $(0, 0, \dots 0, 0)$, the nearest points (distant $\sqrt{2}$) are $(1^2, 0^{m-2})$, namely the vertices of $t_1\alpha_{m-1}\sqrt{2}$.

(iii) $t_1\alpha_{m-1}\sqrt{2}$ has just two kinds of bounding figures:

$$\begin{cases} t_1\alpha_{m-2}\sqrt{2}, & \text{with vertices} \quad (1^2, 0^{m-3}; 0), \\ \alpha_{m-2}\sqrt{2}, & \quad ,, \qquad ,, \qquad (1; 1, 0^{m-2}). \end{cases}$$

These points, along with A, occur among the vertices of

$$\begin{cases} h\gamma_{m-1}\sqrt{2}: & (1^{2r}, 0^{m-2r-1}; 0); \qquad r = 0, 1, \dots \left[\frac{m-1}{2}\right], \\ \alpha_{m-1}\sqrt{2}: & (0^m), \qquad (1; 1, 0^{m-2}), \end{cases}$$

respectively. But $h\gamma_3$ is uniform. Hence, by induction, $h\gamma_m$ is uniform; its vertex figure being

$$t_1\alpha_{m-1}.$$

By subtracting *one* of the co-ordinates from unity, it is clear that the rest of the points 6.21, namely

6.23 $$(1^{2r+1}, 0^{m-2r-1}); \qquad r = 0, 1, \dots \left[\frac{m-1}{2}\right],$$

are the vertices of the complementary $h\gamma_m\sqrt{2}$.

6.3. By 3.81 and 5.28, the numerical properties of $t_1\alpha_{m-1}$ are

$({}^0\|_{m-1})$	$({}^1\|_{m-1})$	$({}^{s-1}\|_{m-1})$ $(s > 2)$	
$\binom{m}{2}$	$3\binom{m}{3}$	$\binom{m}{s}$ +	$(s+1)\binom{m}{s+1}$
		$t_1\alpha_{s-1}$	α_{s-1}

The properties of $h\gamma_m$ can now be deduced by means of 2.52, since we know that

$$({}^0|_m) = 2^{m-1}.$$

Making " $({}^{s-1}|_{m-1,\,1})$ " refer to $t_1\alpha_{m-1}$, the results are as follows :—

$({}^0\|_m)$	$({}^1\|_m)$	$({}^2\|_m)$	$({}^s\|_m)$ $(s > 2)$	
2^{m-1}	$2^{m-2}\binom{m}{2}$	$2^{m-1}\binom{m}{3}$	$2^{m-s}\binom{m}{s}$ +	$2^{m-1}\binom{m}{s+1}$
		α_2	$h\gamma_s$	α_s

Putting $s = m - 1$, we see that (if $m > 3$) $h\gamma_m$ is bounded by

$$2m\ h\gamma_{m-1}\text{'s} \quad \text{and} \quad 2^{m-1}\ \alpha_{m-1}\text{'s}.$$

On referring to the co-ordinates, it is found that the centres of the bounding $h\gamma_{m-1}$'s and of the bounding α_{m-1}'s are the vertices of $\beta_m\times$ and $h\gamma_m\times$ respectively. This is a particular case of the phenomenon called " semi-reciprocation," explained in the next chapter (7.8).

6.4. By 3.9, the circum-radius of γ_m (and therefore of $h\gamma_m\sqrt{2}$) is $\frac{1}{2}\sqrt{m}$. Hence that of $h\gamma_m$ must be

$${}_0\mathrm{R}_m = \tfrac{1}{2}\sqrt{\frac{m}{2}}.$$

The order of the group of symmetries is given by 2.41 and 5.6 :

$$g_m = 2^{m-1}\,.\,(1 + \varepsilon_{(m-1)\,3})\,m\,!$$
$$= (1 + \varepsilon_{m4})\,2^{m-1}\,m\,!\,.$$

Except when $m = 4$, this order is, as we should expect, half that of γ_m.

6.5. The vertices of δ_m are (by 3.6) the points

$$(x_1, x_2, \dots x_{m-1})$$

whose co-ordinates are every possible set of $m - 1$ integers. These points fall into two categories, according as their co-ordinates have an even or odd sum. The points in either category are the vertices of a degenerate polytope called

$$h\delta_m \sqrt{2}.$$

Let us apply 6.1 to the former set—

6.51 $$(x_1, x_2, \ldots x_{m-1});\quad x_1 + x_2 + \ldots + x_{m-1} = 0 \text{ (mod. 2)}.$$

(i) These points are equivalent, by means of the operation of adding 1 to each of two co-ordinates.

(ii) Taking A at the origin (0^{m-1}), the nearest points are $\pm (1^2, 0^{m-3})$, namely the vertices of $t_1\beta_{m-1}\sqrt{2}$.

(iii) $t_1\beta_{m-1}\sqrt{2}$ has two kinds of bounding figures :

$$\begin{cases} t_1\alpha_{m-2}\sqrt{2}, & \text{with vertices } (1^2, 0^{m-3}), \\ \beta_{m-2}\sqrt{2}, & \text{with vertices } (1;\ \pm 1, 0^{m-3}). \end{cases}$$

These points, along with A, occur among the vertices of

$$\begin{cases} h\gamma_{m-1}\sqrt{2}: & (1^{2r}, 0^{m-2r-1});\quad r = 0, 1, \ldots \left[\frac{m-1}{2}\right], \\ \beta_{m-1}\sqrt{2}: & (0^{m-1}), (1;\ \pm 1, 0^{m-3}),\quad (2;\ 0^{m-2}), \end{cases}$$

respectively. Hence $h\delta_m$ is uniform ; its vertex figure being

$$t_1\beta_{m-1}.$$

6.6. Since, by 5.5, $t_1\beta_4 = \{3, 4, 3\}$, it follows that

$$h\delta_5 = \{3, 3, 4, 3\}.$$

Note also

$$\begin{cases} h\delta_3 = \delta_3, \\ h\delta_2 = \delta_2\sqrt{2}. \end{cases}$$

$h\delta_4$ (the system of tetrahedra and octahedra filling three-dimensional space) is (by 1.9) "super-Archimedean," as also are

$$\begin{cases} t_1\beta_3 & \text{(the cuboctahedron, vertex figure of } h\delta_4), \\ t_1\{3, 5\} & \text{(the icosidodecahedron)}, \\ t_1\{3, 6\} & \text{(the system of triangles and hexagons, two and two at each vertex, filling a plane)}. \end{cases}$$

Since $t_1\beta_{m-1}$ is bounded by 2^{m-1} $t_1\alpha_{m-2}$'s and $2(m-1)$ β_{m-2}'s, it follows that $h\delta_m$ has 2^{m-1} $h\gamma_{m-1}$'s and $2(m-1)$ β_{m-1}'s meeting at each vertex. On referring to the co-ordinates, it is found that the centres of the $h\gamma_{m-1}$'s and of the β_{m-1}'s are the vertices of $\delta_m\times$ and $h\delta_m$ respectively.

6.7. Since $h\gamma_m$ and $h\delta_m$ have uniform *second* vertex figures ($[\alpha_1, \alpha_{m-3}]$ and $[\alpha_1, \beta_{m-3}]$ respectively), it follows from 5.8 that they each have *two* simple truncations.

$$\begin{cases} t_1 h\gamma_m & \text{has vertex figure} \quad [\alpha_1, \alpha_1, \alpha_{m-3}] = [\beta_2, \alpha_{m-3}], \\ t_2 h\gamma_m & \text{,,} \quad \text{,,} \quad [\alpha_2, (\alpha_0 \overset{}{\underset{\sqrt{2}}{\text{———}}} \alpha_{m-4})], \\ t_1 h\delta_m & \text{,,} \quad \text{,,} \quad [\alpha_1, \alpha_1, \beta_{m-3}] = [\beta_2, \beta_{m-3}], \\ t_2 h\delta_m & \text{,,} \quad \text{,,} \quad [\alpha_2, (\alpha_0 \underset{\sqrt{2}}{\text{———}} \beta_{m-4})]. \end{cases}$$

On comparing with 5.6, we thus find that

$$\begin{cases} t_1 h\gamma_m = t_2\gamma_m, \\ t_1 h\delta_m = t_2\delta_m. \end{cases}$$

6.8. Let

$$\alpha_m h\sqrt{2} \qquad (\text{“}\alpha_m\text{-hedroid”})$$

denote the section of δ_{m+2} or of $h\delta_{m+2}\sqrt{2}$ by the m-space

$$x_1 + x_2 + \ldots + x_m + x_{m+1} = 0,$$

that is, the degenerate $(m+1)$-dimensional polytope whose vertices are the points

6.81 $$(x_1, x_2, \ldots x_{m+1}); \qquad x_1 + x_2 + \ldots + x_{m+1} = 0.$$

Of these points, those nearest to (*i.e.*, distant $\sqrt{2}$ from) the typical point (0^{m+1}) are

6.82 $$(1, 0^{m-1}, -1).$$

The m-dimensional polytope whose vertices are the $m(m+1)$ points 6.82 will be called

$$e\alpha_m\sqrt{2} \qquad (\text{“expanded } \alpha_m\text{”}).$$

Of these points, those nearest to $(1;0^{m-1};-1)$ are

$$(0;\ 1, 0^{m-2};\ -1) \quad \text{and} \quad (1;\ 0^{m-2}, -1;\ 0),$$

namely the vertices of an $(m-1)$-dimensional polytope which may be described as an "antiprism" on $\alpha_{m-2}\sqrt{2}$ as base. This antiprism (when reduced in linear dimensions by $1:\sqrt{2}$) is denoted by

6.83 $$(\alpha_{m-2} \underset{\sqrt{2}}{=\!=\!=} \alpha_{m-2}).$$

It is bounded by two α_{m-2}'s, reciprocally situated in parallel $(m-2)$-spaces, together with $\binom{m-1}{n}$ $(\alpha_{n-1} \underset{\sqrt{2}}{\text{——}} \alpha_{m-n-2})$'s each joining an α_{n-1} of the first α_{m-2} to the reciprocally corresponding α_{m-n-2} of the second, for all relevant values of n.

By considering the points

6.84 $$(1, 0^n;\ 0^{m-n-1}, -1),$$

we see that $e\alpha_m$ is bounded by $\binom{m+1}{n+1}$ prisms $[\alpha_n, \alpha_{m-n-1}]$, for all values of n from 0 to $m-1$. Therefore $e\alpha_m$ is uniform, its vertex figure being the antiprism 6.83.

Hence also the $(s-1)$-dimensional elements of $e\alpha_m$ consist of $\binom{m+1}{s+1}\binom{s+1}{n+1}$ prisms $[\alpha_n, \alpha_{s-n-1}]$, for all values of n from 0 to $s-1$.

Now, the vertices 6.84 of a typical bounding figure of the $e\alpha_m\sqrt{2}$ (vertices 6.82) occur, along with (0^{m+1}), among the points

$$(1^r, 0^{n-r+1};\ 0^{m-n-r}, -1^r);\ r = 0, 1, \ldots \text{min.}\ (n+1, m-n).$$

These points are seen to be the vertices of the $t_n\alpha_m\sqrt{2}$ obtained from $(1^{n+1}, 0^{m-n})$ by subtracting 1 from each of the first $n+1$ co-ordinates and then reversing all the signs. It follows by 6.1 that $\alpha_m h$ is uniform. The vertex figure of $\alpha_m h$ is $e\alpha_m$, and its bounding figures consist of all the simple truncations of α_m, each vertex being surrounded by

$$\binom{m+1}{n+1}\quad t_n\alpha_m\text{'s}\quad (0 \leq n \leq m-1).$$

Also the s-dimensional elements of $\alpha_m h$ at one vertex consist of

$$\binom{m+1}{s+1}\binom{s+1}{n+1}\quad t_n\alpha_s\text{'s}\qquad (0 \leq n \leq s-1).$$

The figure obtained by drawing sphere-analogues (of unit diameter) with centres at all the vertices of $\alpha_m h$, seems to represent the closest possible packing of an infinity of rigid sphere-analogues in m dimensions. (The three-dimensional case is known as "normal piling.") The number of sphere-analogues which touch a given one is thus $m(m+1)$, the number of vertices of $e\alpha_m$.

Since $\alpha_m h$ possesses a second vertex figure, it has (by 5.8) a truncation

$$t_1\alpha_m h,$$

bounded by $e\alpha_m$'s and $t_1t_n\alpha_m$'s, whose vertex figure is

$$[\alpha_1, (\alpha_{m-2} \overset{}{\underset{\sqrt{2}}{=\!=\!=}} \alpha_{m-2})].$$

6.9. $e\alpha_m$ can be constructed as follows.* Take α_m (supposed of unit edge), and move all its bounding α_{m-1}'s symmetrically away from its centre, each through a distance equal to the circum-radius of α_m. Two α_{m-1}'s which were originally adjacent are now separated to such an extent that their bounding α_{m-2}'s, one of each, which originally coincided, now appear in parallel $(m-2)$-spaces at unit distance apart. These two α_{m-2}'s can be connected by a prism $[\alpha_{m-2}, \alpha_1]$. The new polytope is still not completely

* This construction is due to MRS. BOOLE STOTT (see Preface). The "e" of $e\alpha_m$ is short for her "e_{m-1}."

bounded until we have inserted prisms $[\alpha_{m-3}, \alpha_2]$, ... $[\alpha_1, \alpha_{m-2}]$, and finally α_{m-1}'s (corresponding to the old vertices).

Since the same $e\alpha_m$ can be constructed from the reciprocal α_m, $e\alpha_m$ has twice as many symmetries as α_m. Thus

$$g_m = 2\,(m+1)\,! \qquad (m > 1).$$

This result can also be obtained by 2.41 ; since

$$({}^0|_m) = m\,(m+1),$$

while the antiprism 6.83 possesses the $(m-1)\,!$ symmetries of α_{m-2} combined with the reflection in its own centre.

Note the following particular cases :—

$e\alpha_3 = t_1\beta_3$,	$\alpha_3 h = h\delta_4$,
$e\alpha_2 = \{6\}$,	$\alpha_2 h = \{3, 6\}$,
$e\alpha_1 = \beta_1\sqrt{2}$,	$\alpha_1 h = \delta_2$.

7. Π_r^{+u} *and* n_{pq}.

7.1. Let

$$\Pi_r^{+1}$$

denote the uniform $(r+1)$-dimensional polytope (if such exists) whose vertex figure is a given (finite) polytope Π_r, and

$$\Pi_r^{+u}$$

the uniform $(r+u)$-dimensional polytope (if there is one) whose vertex figure is Π_r^{+u-1}.

It follows from this definition, that the nth vertex figure of Π_r^{+u} is

7.11 $$\Pi_r^{+u-n},$$

Π_r^{+0} being the same as Π_r.

The particular cases when Π_r is regular are as follows :—

$(\alpha_1\, 2\cos\pi/k)^{+1} = \{k\}$,	$(\alpha_1\sqrt{3})^{+2} = \{3, 6\}$,
$\alpha_r^{+u} = \alpha_{r+u}$,	$(\alpha_2\sqrt{3})^{+1} = \{6, 3\}$,
$\beta_r^{+u} = \beta_{r+u}$,	$(\alpha_1\tau)^{+3} = \{3, 3, 5\}$,
$(\alpha_r\sqrt{2})^{+1} = \gamma_{r+1}$,	$(\alpha_3\tau)^{+1} = \{5, 3, 3\}$,
$(\beta_r\sqrt{2})^{+1} = \delta_{r+1}$,	$(\alpha_2\sqrt{2})^{+2} = \gamma_3^{+1} = \{3, 4, 3\}$,
$(\alpha_1\tau)^{+2} = \{3, 5\}$,	$(\alpha_2\sqrt{2})^{+3} = \gamma_3^{+2} = \{3, 3, 4, 3\}$,
$(\alpha_2\tau)^{+1} = \{5, 3\}$,	$(\alpha_3\sqrt{2})^{+2} = \gamma_4^{+1} = \{3, 4, 3, 3\}$.

Note that $(\alpha_r a)^{+u}$ and $(\alpha_u a)^{+r}$ are reciprocal.

As a further example of the notation, $(e\alpha_m)^{+1} = \alpha_m h$.

Assuming Π_r^{+1} to have (by definition) unit edges (α_1), we can assert that every u-dimensional element of Π_r^{+u} is

$$\Pi_0^{+u} = (\Pi_0^{+1})^{+u-1} = (\alpha_1)^{+u-1} = \alpha_u.$$

Making the convention that

7.12 $$\Pi_{n-u}^{+u} = \alpha_n \quad \text{if} \quad n < u,$$

it follows that *every* n-dimensional element of Π_r^{+u} is of the form

7.13 $$\Pi_{n-u}^{+u}.$$

In particular, the bounding figures are of the form Π_{r-1}^{+u}.

7.2. In 5.8 we remarked that Π_m has an nth truncation if it has an $(n+1)$th vertex figure. This condition is satisfied if $\Pi_m = \Pi_r^{+n+1}$, since then $\Pi_{m-n-1,\, n+1} = \Pi_r$.

By 5.31, the vertex figure of $t_n \Pi_r^{+n+1}$* is $[(\Pi_{-1}^{+n+1})', \Pi_r]$, *i.e.*, $[\alpha_n, \Pi_r]$. Thus we may write

7.21 $$t_n \Pi_r^{+n+1} = [\alpha_n, \Pi_r]^{+1}.$$

By 3.9 and 4.29, the squared circum-radius of $[\alpha_n, \Pi_r]$ is

$$\tfrac{1}{2}\left(1 - \frac{1}{n+1}\right) + ({}_0\mathrm{R}_r)^2.$$

Hence, by 2.84, Π_r^{+n+1} *cannot* exist if

$$\tfrac{1}{2}\left(1 - \frac{1}{n+1}\right) + ({}_0\mathrm{R}_r)^2 > 1,$$

i.e., if

7.22 $$1 + \frac{1}{n+1} < 2\,({}_0\mathrm{R}_r)^2.$$

By 2.83, it can only be degenerate in the critical case when

$$1 + \frac{1}{n+1} = 2\,({}_0\mathrm{R}_r)^2,$$

since then $[\alpha_n, \Pi_r]$ must have *unit* circum-radius.

The inequality 7.22 can alternatively be obtained as follows. By 2.83 and 2.81,

$${}_0\mathrm{R}_m = \sqrt{\tfrac{1}{2}\left(1 + \frac{1}{x-1}\right)} \quad \text{if} \quad {}_0\mathrm{R}_{m-1,\,1} = \sqrt{\tfrac{1}{2}\left(1 + \frac{1}{x}\right)}.$$

* Meaning $t_n(\Pi_r^{+n+1})$ and not $(t_n \Pi_r)^{+n+1}$.

Hence, if Π_r has circum-radius ${}_0R_r = \sqrt{\tfrac{1}{2}\left(1+\frac{1}{x}\right)}$,

then Π_r^{+u} has circum-radius $\sqrt{\tfrac{1}{2}\left(1+\frac{1}{x-u}\right)}$,

which is imaginary if $x < u < x+1$.

So Π_r^{+u} is impossible if $x < u$,*

i.e., if $1+\frac{1}{u} < 2\,({}_0R_r)^2$.

It is degenerate in the case of equality, because its circum-radius is then infinite.

7.3. The main object of this paper is to examine all polytopes of the form

$$[\Pi_{m_1}^{(1)},\ \Pi_{m_2}^{(2)}]^{+u},$$

the constituents being regular. With these are intimately associated the polytopes of the form

7.31 $$[\Pi_{m_1}^{(1)},\ \Pi_{m_2}^{(2)},\ \Pi_{m_3}^{(3)}]^{+1}.$$

But $[\Pi_{m_1}^{(1)}, \Pi_{m_2}^{(2)}, \Pi_{m_3}^{(3)}]^{+2}$ and $[\Pi_{m_1}^{(1)}, \Pi_{m_2}^{(2)}, \Pi_{m_3}^{(3)}, \Pi_{m_4}^{(4)}]^{+1}$ never occur; because $[\alpha_2, \alpha_2, \alpha_1]^{+1}$ and $[\alpha_2, \alpha_2, \alpha_2, \alpha_1]$ have circum-radii $\sqrt{3}$ and $\sqrt{\frac{5}{4}}$ respectively, while *a fortiori* more complicated would-be vertex figures have circum-radii exceeding unity.

It is easily verified that the only possible polytopes of the form

$$[\Pi_{m_1}^{(1)},\ \Pi_{m_2}^{(2)}]^{+1},$$

with the existence-condition (by 2.84 and 4.29)

$$({}_0R_{m_1}^{(1)})^2 + ({}_0R_{m_2}^{(2)})^2 \leq 1,$$

* For, if $u \geqslant x+1$,

let $y = [u-x]$.

Then $u-x \geqslant y > u-x-1$,

i.e., $x \leqslant u-y < x+1$.

Hence Π_r^{+u-y} is degenerate or impossible.

But $y \geqslant 1$.

Therefore Π_r^{+u} is still impossible.

are the simple truncations of the regular polytopes, namely :—

$$[\alpha_p, \alpha_q]^{+1} = t_p\,\alpha_{p+q+1} = t_q\,\alpha_{p+q+1},$$
$$[\alpha_p, \beta_q]^{+1} = t_p\,\beta_{p+q+1} = t_q\,\gamma_{p+q+1},$$
$$[\beta_p, \beta_q]^{+1} = t_p\,\delta_{p+q+1} = t_q\,\delta_{p+q+1},$$
$$[\alpha_p, \alpha_q\sqrt{2}]^{+1} = t_p\,\gamma_{q+1}^{+p} = t_q\,\gamma_{p+2}^{+q-1} \quad \left(\frac{1}{p+1}+\frac{2}{q+1} \geq 1\right),$$
$$[\alpha_p, (\alpha_1\tau)^{+u}]^{+1} = t_p\,(\alpha_1\tau)^{+p+u+1} = t_{u+1}\,(\alpha_{p+u+1}\tau)^{+1} \quad (p+u \leq 2^*),$$
$$[\alpha_1, \alpha_1\sqrt{3}]^{+1} = t_1\,(\alpha_1\sqrt{3})^{+2} = t_1\,(\alpha_2\sqrt{3})^{+1}.$$

All these, except $[\beta_p, \beta_q]^{+1}$, are particular cases of

$$[\alpha_p, \Pi_q]^{+1} = t_p\Pi_q^{+p+1},$$

which is the same as 7.21.

By 5.6, the only truncations with circum-radii ≤ 1 are:

$$[\alpha_p, \alpha_q]^{+1} \quad \text{with} \quad \frac{1}{p+1}+\frac{1}{q+1} \geq \tfrac{1}{2},$$

and

$$[\alpha_1, \beta_q]^{+1}.$$

Now

$$[\alpha_1, \beta_q]^{+2} = (t_1\beta_{q+2})^{+1} = h\delta_{q+3}$$

is degenerate, and so cannot be a vertex figure.

We are thus naturally led to consider all possible polytopes of the special form

$$[\alpha_p, \alpha_q]^{+u},$$

or more conveniently

$$[\alpha_p, \alpha_q]^{+n+1},$$

for which, by 7.22, the existence-condition is

$$1+\frac{1}{n+1} \geq \left(1-\frac{1}{p+1}\right)+\left(1-\frac{1}{q+1}\right),$$

i.e.,

7.32 $$\frac{1}{n+1}+\frac{1}{p+1}+\frac{1}{q+1} \geq 1$$

(equality indicating degeneracy).

* The circum-radius of $\alpha_1\tau$ being $\tfrac{1}{2}\tau = \sqrt{\tfrac{1}{2}\left(1+\frac{1}{2\tau}\right)}$,

that of $(\alpha_1\tau)^{+u}$ must be $\sqrt{\tfrac{1}{2}\left(1+\frac{1}{2\tau-u}\right)}$.

So $(\alpha_1\tau)^{+u}$ is impossible if $u > 2\tau$;

and $[\alpha_p, (\alpha_1\tau)^{+u}]^{+1}$, if $\tfrac{1}{2}\left(1-\frac{1}{p+1}\right)+\tfrac{1}{2}\left(1+\frac{1}{2\tau-u}\right) > 1$,

i.e., if $p+u > 2\tau-1 \;(=\sqrt{5})$.

The symmetrical character of this condition suggests the new notation

7.33 $$n_{pq} = [\alpha_p, \alpha_q]^{+n+1},$$

the p and q being of course interchangeable, so that

7.34 $$n_{pq} = n_{qp}.$$

Putting $[\alpha_p, \alpha_q]$ for Π_r in **7.21**, we obtain the identity

7.35 $$t_n n_{pq} = [\alpha_n, \alpha_p, \alpha_q]^{+1},$$

from which it follows that

7.36 $$t_n n_{pq} = t_p p_{qn} = t_q q_{np}.$$

Three polytopes, n_{pq}, p_{qn}, q_{np}, which are related in this manner, are said to be "semi-reciprocals" of one another.

7.4. The following are the special cases of **7.33** so far discussed:

7.41 $$(-1)_{pq} = [\alpha_p, \alpha_q],$$

7.42 $$0_{pq} = t_p \alpha_{p+q+1}, \qquad p_{q0} = \alpha_{p+q+1}, \qquad q_{0p} = \alpha_{p+q+1},$$

7.43 $$n_{11} = \beta_{n+3}, \qquad 1_{1n} = h\gamma_{n+3}, \qquad 1_{n1} = h\gamma_{n+3}.$$

To these might (by analogy) be added

$$(-2)_{pq} = (\alpha_{p-1} \underset{\sqrt{2}}{\overline{\qquad}} \alpha_{q-1}).$$

The only remaining possibilities, according to **7.32**, are:—

7.45

	2_{21}		1_{22}
3_{21}		2_{31}	1_{32}
4_{21}		2_{41}	1_{42}
5_{21}		2_{51}	1_{52}
	3_{31}		1_{33}
		2_{22}	

The existence of these fourteen polytopes remains to be established (in chapter 9). But let us first investigate their properties on the assumption that they do exist. Note that the last six of them satisfy the degeneracy-condition

7.46 $$\frac{1}{n+1} + \frac{1}{p+1} + \frac{1}{q+1} = 1.$$

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7.5. By 7.11, the uth vertex figure of

$$n_{pq}$$

is

$$(n-u)_{pq}.$$

The *elements* of n_{pq} fall into two categories: those of $\leq n$ dimensions, which are all α's (simplexes) by **7.12**; and those of $> n$ dimensions, which, by **7.13**, are all of the form

$$n_{p'q'}$$

where

$$0 \leq p' \leq p \quad \text{and} \quad 0 \leq q' \leq q.$$

In spite of 7.34, it is useful to fix the order of the suffixes, so as to distinguish between equal elements which are of different type. Thus, if $p > q \geq r > s \geq 0$, we say n_{pq} has elements n_{rs} of two different types,

$$n_{rs} \quad \text{and} \quad n_{sr}.$$

If $s > 0$, an element of type $n_{r(s-1)}$ belongs to an element of type n_{rs}, but cannot belong to one of type n_{sr}, since $r > s$.

We can prove by induction, proceeding as in 3.2, that all elements of the same type are equivalent, this being obvious when $n = -1$. Equal elements of different type are not equivalent, unless $p = q$; and even then it is worth while to preserve the distinction, such elements being (like the faces 0_{10} and 0_{01} of the octahedron 0_{11}) uniquely divisible into two congruent sets.

7.6. Let

$$[n\ p\ q]$$

denote the number of vertices of $[\alpha_n, \alpha_p, \alpha_q]^{+1}$. (Its value is thus independent of the order in which n, p, q occur.) By 7.35, it is also the number of α_n's in n_{pq}, so that (by 7.41)

7.61 $$[(-1)\ p\ q] = 1.$$

By 7.42 and 7.43 respectively,

7.62 $$[0\ p\ q] = \binom{p+q+2}{p+1}$$

and

7.63 $$[n\ 1\ 1] = 2^{n+1}\binom{n+3}{2} = 2^n(n+2)(n+3),$$

Also, by 7.46,

7.64 $$[5\ 2\ 1] = [3\ 3\ 1] = [2\ 2\ 2] = \infty.$$

Thus the only cases which still await calculation are

7.65 $$[2\ 2\ 1], \quad [3\ 2\ 1], \quad [4\ 2\ 1].$$

(These numbers will be found, by means of indeterminate equations, in the next chapter, 8.7.)

7.7. Applying 2.63 (with n for s) to n_{pq}, for which

7.71 $$m = n + p + q + 1,$$

we have

$$({}^{u-1}|_m)\,[(n-u)\,p\,q] = \binom{n+1}{u}\,[n\,p\,q].$$

So the number of α_{u-1}'s of the first category (7.5) is

$$({}^{u-1}|_m) = \binom{n+1}{u}\frac{[n\,p\,q]}{[(n-u)\,p\,q]} \qquad (0 \leq u \leq n+1).$$

More symmetrically, the number of $\alpha_{n-n'-1}$'s is

7.72 $$({}^{n-n'-1}|_m) = \binom{n+1}{n'+1}\frac{[n\,p\,q]}{[n'p\,q]} \qquad (-1 \leq n' \leq n).$$

Again, putting $n + p' + q' + 1$ for s and $n + 1$ for u in 2.63,

$$[n\,p\,q]\,.\,\binom{p+1}{p'+1}\binom{q+1}{q'+1} = [n\,p'q']\,({}^{n+p'+q'+1}|_m).$$

So the number of elements of type

$$n_{p'q'}$$

is

7.73 $$({}^{n+p'+q'+1}|_m) = \binom{p+1}{p'+1}\binom{q+1}{q'+1}\frac{[n\,p\,q]}{[n\,p'q']} \qquad (0 \leq p' \leq p, \quad 0 \leq q' \leq q).$$

Putting $n' = n - 1$ in 7.72, the number of vertices of n_{pq} is

7.74 $$({}^{0}|_m) = (n+1)\frac{[n\,p\,q]}{[(n-1)\,p\,q]}.$$

Again, putting $p' = p - 1$, $q' = q$, and then $p' = p$, $q' = q - 1$, in **7.73**, the number of bounding figures

7.75 $$n_{(p-1)\,q} \quad \text{and} \quad n_{p\,(q-1)}$$

is

7.76 $$({}^{m-1}|_m) = (p+1)\frac{[n\,p\,q]}{[n\,(p-1)\,q]} + (q+1)\frac{[n\,p\,q]}{[n\,p\,(q-1)]}.$$

7.74 and 7.76 reveal the interesting fact that the numbers of vertices and of bounding figures of the two types, take the same values (in different order) for three semi-reciprocal polytopes. This fact naturally suggests, as a theorem worthy of consideration, that the centres of the bounding figures of n_{pq} are the vertices of polytopes similar to p_{qn} and q_{np}. Another way of saying this, is that the reciprocal of n_{pq} has the vertices of $p_{qn}\times$ and $q_{np}\times$; hence the name "semi-reciprocal." This fails when $pq = 0$, because

n_{0q} ($= \alpha_{q+n+1}$) has only one type of bounding figure. But the theorem can hold even in this case, if we make the convention

7.77 $$n_{(-1)q} = \alpha_n$$

(which agrees with **7.36**, **7.42**, **7.61**, **7.72** and **7.74**, though violating **7.71**), and re-enunciate it in the following form.

7.8. *The centres of the* $n_{(p-1)q}$*'s of* n_{pq} *are the vertices of* $p_{qn} \times$.

Let this theorem (which we shall prove) be denoted by

$$(n,\ p,\ q).$$

In the first place, the particular cases

$$(0, p, q), \quad (n, 0, q), \quad (n, p, 0)$$

are obvious, since they state respectively, that

7.81 the centres of the $t_{p-1}\,\alpha_{p+q}$'s of $t_p\alpha_{p+q+1}$ are the vertices of $\alpha_{p+q+1} \times$,

7.82 the centres of the α_n's of α_{q+n+1} are the vertices of $t_n\,\alpha_{q+n+1} \times$,

7.83 the centres of the α_{n+p}'s of α_{n+p+1} are the vertices of $\alpha_{n+p+1} \times$.

(**7.81** is true by **5.22**, **7.82** by the definition of truncation, and **7.83** by reciprocation.) From now on, therefore, we shall suppose

$$n > 0, \qquad p > 0, \qquad q > 0.$$

By the principle of induction, it will be sufficient to deduce (n, p, q) from the theorems

$$(n',\ p',\ q')$$

in which $0 \leq n' \leq n$, $0 \leq p' \leq p$, $0 \leq q' \leq q$, but $n' + p' + q' < n + p + q$.

For the purposes of the proof, we actually hypothesize

7.84 $$\begin{cases} (n-1, p, q), \quad (n, p-1, q), \quad (n, p, q-1), \quad (n-1, p, q-1), \\ (n-2, p, q) \text{ (if } n > 1) \quad \text{and} \quad (n, p, q-2) \text{ (if } q > 1). \end{cases}$$

Two bounding figures (of given types) of n_{pq} are said to be "adjacent" if their contact is the closest possible.

$n_{(p-1)q}$ and $n_{p(q-1)}$ (two bounding figures of different type) are adjacent if, and only if, they have a common $n_{(p-1)(q-1)}$. For, $n_{(p-1)(q-1)}$ occurs as a bounding figure both of $n_{(p-1)q}$ and of $n_{p(q-1)}$; while $n_{(p-2)q}$, the other type of bounding figure of $n_{(p-1)q}$, cannot belong to $n_{p(q-1)}$, nor $n_{p(q-2)}$ to $n_{(p-1)q}$. Further, since all elements of type

$n_{(p-1)(q-1)}$ are equivalent, *every* $n_{(p-1)(q-1)}$ of n_{pq} belongs just to one $n_{(p-1)q}$ and to one $n_{p(q-1)}$.

We shall now prove that the common element of two adjacent $n_{(p-1)q}$'s (bounding figures of the *same* type) is $n_{(p-2)q}$. This is obvious when $p > 1$; since $n_{(p-2)q}$ is a bounding figure of $n_{(p-1)q}$, whereas $n_{(p-1)(q-1)}$ cannot belong to *two* $n_{(p-1)q}$'s. When $p = 1$, what we have to prove is that the common element of two adjacent α_{q+n+1}'s of n_{1q} is α_n. Now, the $(n+1)$th vertex figure of n_{1q} is $[\alpha_1, \alpha_q]$, which has two bounding α_q's. But the $(n+1)$th vertex figure always indicates the incidences at an n-dimensional element. Hence just *two* bounding α_{q+n+1}'s of n_{1q} meet at every α_n. (These two α_{q+n+1}'s must be adjacent, since the two α_q's of $[\alpha_1, \alpha_q]$ are trivially adjacent.) Thus the common element of two adjacent $n_{(p-1)q}$'s of n_{pq} is $n_{(p-2)q}$, even if $p = 1$. Similarly, the common element of two adjacent $n_{p(q-1)}$'s is $n_{p(q-2)}$. Let Π denote the polytope whose vertices are the centres of all the $n_{(p-1)q}$'s of n_{pq}. It has the same number of vertices as p_{qn}: we have to prove that it *is* $p_{qn}\times$.

Let us investigate those vertices of Π which are the centres of certain special sets of $n_{(p-1)q}$'s of n_{pq}. To take the simplest possible set, it is clear that the centres of two adjacent $n_{(p-1)q}$'s are two *consecutive* vertices of Π (*i.e.*, two vertices joined by an edge).

Those $n_{(p-1)q}$'s which are adjacent to a given $n_{p(q-1)}$ meet the latter in its $n_{(p-1)(q-1)}$'s. Hence the centres of these $n_{(p-1)q}$'s are the vertices of a polytope similar to that whose vertices are the centres of the $n_{(p-1)(q-1)}$'s of $n_{p(q-1)}$. By $(n, p, q-1)$, this polytope is

7.85 $$p_{(q-1)n}\times.$$

Again, the centres of those $n_{(p-1)q}$'s which meet at a given vertex of n_{pq} are the vertices of a polytope similar to that whose vertices are the centres of the bounding $(n-1)_{(p-1)q}$'s of the vertex figure $(n-1)_{pq}$. By $(n-1, p, q)$, this polytope is

7.86 $$p_{q(n-1)}\times.$$

From the manner in which they were determined, the $p_{(q-1)n}\times$ and $p_{q(n-1)}\times$, whose vertices are the centres of these two special sets of $n_{(p-1)q}$'s of n_{pq}, are bounding figures of Π. In order to show that such figures *completely* bound Π, we must examine the bounding $p_{(q-2)n}\times$'s and $p_{(q-1)(n-1)}\times$'s of 7.85 and the bounding $p_{(q-1)(n-1)}\times$'s and $p_{q(n-2)}\times$'s of 7.86. If $q = 1$ or $n = 1$, *bounding* $p_{(q-2)n}\times$'s or $p_{q(n-2)}\times$'s (respectively) do not occur.

The centres of those $n_{(p-1)q}$'s of n_{pq} which are adjacent to a given $n_{p(q-1)}$ and also occur at a given vertex of this $n_{p(q-1)}$, are the vertices of a polytope similar to that whose vertices are the centres of the bounding $(n-1)_{(p-1)(q-1)}$'s of $(n-1)_{p(q-1)}$. By $(n-1, p, q-1)$, this polytope is $p_{(q-1)(n-1)}\times$. From the manner of its construction, such a polytope occurs $(n+1)\dfrac{[n\,p\,(q-1)]}{[(n-1)\,p\,(q-1)]}$ times as a bounding figure of 7.85

(*viz.*, once for every vertex of $n_{p(q-1)}$), and $(q+1)\dfrac{[(n-1)\,p\,q]}{[(n-1)\,p\,(q-1)]}$ times as a bounding figure of 7.86 (*viz.*, once for every $n_{p(q-1)}$ at a vertex of n_{pq}). Thus *every* $p_{(q-1)(n-1)}\times$ which belongs to a $p_{(q-1)n}\times$ or $p_{q(n-1)}\times$ of Π belongs also to a $p_{q(n-1)}\times$ or $p_{(q-1)n}\times$ respectively.

If $q>1$, those $n_{(p-1)q}$'s of n_{pq} which are adjacent to both of two given adjacent $n_{p(q-1)}$'s, meet the common $n_{p(q-2)}$ of these $n_{p(q-1)}$'s in its $n_{(p-1)(q-2)}$'s. Hence the centres of these $n_{(p-1)q}$'s are the vertices of a polytope similar to that whose vertices are the centres of the $n_{(p-1)(q-2)}$'s of $n_{p(q-2)}$. By $(n, p, q-2)$, this polytope is $p_{(q-2)n}\times$. From the manner of its construction, such a polytope occurs $q\dfrac{[n\,p\,(q-1)]}{[n\,p\,(q-2)]}$ times as a bounding figure of 7.85 (*viz.*, once for every $n_{p(q-2)}$ of $n_{p(q-1)}$). Thus *every* $p_{(q-2)n}\times$ which belongs to a $p_{(q-1)n}\times$ of Π belongs also to another $p_{(q-1)n}\times$.

Again, if $n>1$, the centres of those $n_{(p-1)q}$'s of n_{pq} which occur at a given edge, are the vertices of a polytope similar to that whose vertices are the centres of the bounding $(n-2)_{(p-1)q}$'s of the second vertex figure $(n-2)_{pq}$. By $(n-2, p, q)$, this polytope is $p_{q(n-2)}\times$. From the manner of its construction, such a polytope occurs $n\dfrac{[(n-1)\,p\,q]}{[(n-2)\,p\,q]}$ times as a bounding figure of 7.86 (*viz.*, once for every edge at a vertex of n_{pq}). Thus *every* $p_{q(n-2)}\times$ which belongs to a $p_{q(n-1)}\times$ of Π belongs also to another $p_{q(n-1)}\times$.

We have now proved that Π is completely bounded by the aforesaid $p_{(q-1)n}\times$'s and $p_{q(n-1)}\times$'s. Also, the vertices of Π are, like the $n_{(p-1)q}$'s of n_{pq}, equivalent. Hence (by 1.7) Π is uniform. In order to identify it with $p_{qn}\times$, we have only to prove that its vertex figure is

$$(p-1)_{qn}.$$

In order to do this, consider those $n_{(p-1)q}$'s of n_{pq} which are adjacent to a given $n_{(p-1)q}$. These $n_{(p-1)q}$'s meet the given $n_{(p-1)q}$ in its $n_{(p-2)q}$'s. Hence their centres are the vertices of a polytope similar to that whose vertices are the centres of the $n_{(p-2)q}$'s of $n_{(p-1)q}$. By $(n, p-1, q)$, this polytope is $(p-1)_{qn}\times$.

Thus the vertex figure of Π is $(p-1)_{qn}\times$. But this vertex figure must be bounded by $(p-1)_{(q-1)n}$'s and $(p-1)_{q(n-1)}$'s, these being the vertex figures of $p_{(q-1)n}\times$ and $p_{q(n-1)}\times$ respectively. Hence the vertex figure of Π is precisely $(p-1)_{qn}$, and so $\Pi=p_{qn}\times$.

Since $(0, p, q)$, $(n, 0, q)$ and $(n, p, 0)$ are all true, while (n, p, q) can be deduced from 7.84, it follows by induction that (n, p, q) is true for all relevant values of n, p, q (*i.e.*, whenever n_{pq} exists).

This "semi-reciprocation theorem," as it may be called, is only a particular case of a more general theorem, to the effect that *the centres of the $n_{p'q}$'s of n_{pq} are the vertices of* $t_{p-p'-1}\,p_{qn}\times$.

7.9. Let g_m be the order of the group of symmetries of n_{pq}, so that $g_{m-1,1}$ is the order of the group of symmetries of $(n-1)_{pq}$. We shall prove that, if $p+q>0$,*

7.91 $$g_m = (1+\varepsilon_{pq})(n+1)!\,(p+1)!\,(q+1)!\,[n\,p\,q].$$

This is true when $n=-1$, since it becomes 4.72. Also, it can be deduced from

$$g_{m-1,1} = (1+\varepsilon_{pq})\,n!\,(p+1)!\,(q+1)!\,[(n-1)\,p\,q]$$

by means of 2.41 and 7.74.

Hence it is true, by induction.

For the purposes of group-theory, the violation of 7.71 is a fatal defect of the convention 7.77, which must apply only when $q=0$. When n and q are both positive, it is convenient to assume

7.92 $$n_{(-1)q} = \{3, \ldots 3, 2, 3, \ldots 3\}$$

with $n-1$ threes at the beginning and $q-1$ threes at the end. "Improper" regular polytopes like this, whose SCHLÄFLI symbols contain the number 2, are found to have zero content, and are therefore most conveniently regarded as partitions of (the boundary of) a sphere-analogue. When so regarded, they become perfectly analogous to the central projection of a proper finite regular polytope on a concentric sphere-analogue. The simplest example is the "digon"

$$1_{(-1)1},$$

which can be regarded as the partition of (the circumference of) a circle into two semicircles.

According to the new convention 7.92, the elements of $n_{(-1)q}$ consist of

$$({}^{s}|_{m}) = \tbinom{n+1}{s+1} \quad \alpha_s\text{'s}, \quad \text{for} \quad s \leq n-1,$$

and

$$({}^{n+q'}|_{m}) = \tbinom{q+1}{q'+1} \quad n_{(-1)q'}\text{'s}, \quad \text{for} \quad 0 \leq q' \leq q.$$

The chief disadvantage of this convention is that it makes $({}^{n}|_{m}) = q+1$, in disagreement with $[n\,(-1)\,q]=1$ (7.61). (This happens because the n-dimensional elements now belong to the *second* category; instead of the *first*, as in 7.5.)

Note that $n_{(-1)q}$ and $q_{(-1)n}$ are reciprocal.

8. *The Pure Archimedean Series.*

8.1. We shall now investigate certain special cases of the polytope n_{pq}, with a view to evaluating the numbers 7.65.

* As in 4.72, the ε_{pq} has to be omitted if p and q both vanish. In order to cover this exceptional case, 7.91 may be written in the form

$$g_m = (1+\varepsilon_{pq}-\varepsilon_{p0}\,\varepsilon_{q0})\,(n+1)!\,(p+1)!\,(q+1)!\,[n\,p\,q].$$

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The Π_{m-3}'s ($n_{00} = \alpha_{n+1}$) of n_{21} are all equivalent (7.5). So also are the Π_{m-4}'s ($n_{00} = \alpha_{n+1}$) of n_{31}, and the Π_{m-3}'s ($n_{10} = \alpha_{n+2} = n_{01}$) of n_{22}. Further, the Π_{m-2}'s ($n_{10} = \alpha_{n+2} = n_{01}$) of n_{21}, and likewise the Π_{m-3}'s ($n_{10} = \alpha_{n+2} = n_{01}$) of n_{31}, are equal but not equivalent. Also the Π_{m-1}'s ($n_{21} = n_{12}$) of n_{22} are equal (in fact, equivalent), though the Π_{m-2}'s are not (being actually $n_{20} = \alpha_{n+3} = n_{02}$ and $n_{11} = \beta_{n+3}$). For these reasons, in accordance with 1.9, we call the polytopes

$$\begin{cases} n_{21} & (\text{for } -2 \leq n \leq 5) \text{ the "pure Archimedean series,"} \\ n_{31} & (\text{for } -2 \leq n \leq 3) \text{ the "sub-Archimedean series,"} \\ n_{22} & (\text{for } -2 \leq n \leq 2) \text{ the "isohedral Archimedean series";} \end{cases}$$

and adopt the alternative notation

8.11 $$(\mathrm{PA})_{n+4} = n_{21},$$

8.12 $$(\mathrm{SA})_{n+5} = n_{31},$$

8.13 $$(\mathrm{IA})_{n+5} = n_{22}.$$

Thus, *e.g.*,

8.14 $$(\mathrm{PA})_2 = (\alpha_1 \overset{}{\underset{\sqrt{2}}{\text{———}}} \alpha_0), \quad (\mathrm{PA})_3 = [\alpha_2, \alpha_1], \quad (\mathrm{PA})_4 = t_1\alpha_4, \quad (\mathrm{PA})_5 = h\gamma_5;$$

8.15 $$(\mathrm{SA})_3 = (\alpha_2 \overset{}{\underset{\sqrt{2}}{\text{———}}} \alpha_0), \quad (\mathrm{SA})_4 = [\alpha_3, \alpha_1], \quad (\mathrm{SA})_5 = t_1\alpha_5, \quad (\mathrm{SA})_6 = h\gamma_6;$$

8.16 $$(\mathrm{IA})_3 = (\alpha_1 \overset{}{\underset{\sqrt{2}}{\text{———}}} \alpha_1), \quad (\mathrm{IA})_4 = [\alpha_2, \alpha_2], \quad (\mathrm{IA})_5 = t_2\alpha_5.$$

8.2. In each of these series (as in the series of α's and of β's) every polytope (except the last of all) is the vertex figure of the next. $(\mathrm{PA})_2$, the vertex figure of $[\alpha_2, \alpha_1]$, is an isosceles triangle of sides

$$1, \ \sqrt{2}, \ \sqrt{2}.$$

$(\mathrm{SA})_3$ and $(\mathrm{IA})_3$ both have some claim to the title "isosceles tetrahedron," the former being a triangular right pyramid, and the latter (in the language of crystallography) a "rhombic bisphenoid."

The highest members of the series, namely

$$(\mathrm{PA})_9 = 5_{21}, \quad (\mathrm{SA})_8 = 3_{31}, \quad (\mathrm{IA})_7 = 2_{22},$$

are degenerate (by 7.46).

$(\mathrm{SA})_7$ is semi-reciprocal to $(\mathrm{PA})_7$, and $(\mathrm{IA})_6$ to $(\mathrm{PA})_6$. It is therefore desirable to make a special study of the pure Archimedean series.

8.3. By 7.75, $(\mathrm{PA})_m$ is bounded by α_{m-1}'s ($n_{20} = \alpha_{n+3}$ by 7.42) and β_{m-1}'s ($n_{11} = \beta_{n+3}$ by 7.43). It is convenient to let P_m denote the number of α_{m-1}'s, so that, by 7.76 (with $m - 4$ for n),

$$\mathrm{P}_m = 2\,[(m-4)\,2\,1]/\tbinom{m}{3} \qquad (m > 2)$$

and

8.31 $$[n\,2\,1] = \tfrac{1}{12}(n+2)(n+3)(n+4)\,\mathrm{P}_{n+4}.$$

Further, since $(PA)_2$ is bounded by one α_1 and two β_1's, $P_2 = 1$.

Putting $m-1$ for s in 2.63, $({}^{u-1}|_{m})\,({}^{m-u-1}|^{\sigma}_{m-u,\,u}) = ({}^{u-1}|^{\sigma}_{m-1})\,({}^{m-1}|^{\sigma}_{m})$.
Applying this to $(PA)_m$, whose uth vertex figure is $(PA)_{m-u}$, and letting the "σ" refer to the bounding α_{m-1}'s of $(PA)_m$ (and to the corresponding α_{m-u-1}'s of $(PA)_{m-u}$), we have

$$({}^{u-1}|_{m})\,P_{m-u} = \binom{m}{u} P_m \qquad (u \leq m-2).$$

Thus the number of elements α_{u-1} (for $u \leq m-2$) is

8.32 $$({}^{u-1}|_{m}) = \binom{m}{u} P_m/P_{m-u}.$$

In particular, the number of vertices (for $m > 2$) is

8.33 $$({}^{0}|_{m}) = mP_m/P_{m-1}.$$

8.4. The $(m-2)$-dimensional elements of $(PA)_m$ $(= n_{21})$, though all of them α_{m-2}'s, are of two types: those of type "$\alpha\beta$" $(= n_{10})$ each belong to one bounding α_{m-1} $(= n_{20})$ and to one bounding β_{m-1} $(= n_{11})$, while those of type "$\beta\beta$" $(= n_{01})$ each belong to two bounding β_{m-1}'s. This is obviously true when $m = 2$ (*i.e.*, for the isosceles triangle whose sides are α_1, β_1, β_1) and follows for greater m since $(PA)_2$ is the $(m-2)$th vertex figure of $(PA)_m$.

If $({}^{m-2}|^{\alpha\beta}_{m})$ and $({}^{m-2}|^{\beta\beta}_{m})$ are the numbers of α_{m-2}'s of these two types, while $({}^{m-1}|^{\alpha}_{m})$ and $({}^{m-1}|^{\beta}_{m})$ are the numbers of bounding α's and β's, we have the following relations:

8.41 $$m\,({}^{m-1}|^{\alpha}_{m}) = ({}^{m-2}|^{\alpha\beta}_{m})$$

(since α_{m-1} is bounded by m α_{m-2}'s) and

8.42 $$2^{m-1}({}^{m-1}|^{\beta}_{m}) = ({}^{m-2}|^{\alpha\beta}_{m}) + 2({}^{m-2}|^{\beta\beta}_{m})$$

(since β_{m-1} is bounded by 2^{m-1} α_{m-2}'s).

Putting $s = u = m-2$ in 2.63, $({}^{m-3}|_{m})\,({}^{0}|^{\sigma}_{2,\,m-2}) = ({}^{m-3}|_{m-2})\,({}^{m-2}|^{\sigma}_{m})$.
This can be applied to $(PA)_m$, the "σ" standing for either of the type-symbols $\alpha\beta$, $\beta\beta$. It follows that the obvious relation

$$({}^{0}|^{\alpha\beta}_{2,\,m-2}) = 2\,({}^{0}|^{\beta\beta}_{2,\,m-2})$$

implies

8.43 $$({}^{m-2}|^{\alpha\beta}_{m}) = 2\,({}^{m-2}|^{\beta\beta}_{m}).$$

From

8.44 $$({}^{m-1}|^{\alpha}_{m}) = P_m$$

(the definition of P_m), we can now deduce successively:

8.45 $$({}^{m-2}|^{\alpha\beta}_{m}) = mP_m \qquad \text{(by 8.41)},$$

8.46 $$({}^{m-2}|^{\beta\beta}_{m}) = mP_m/2 \qquad \text{(by 8.43)},$$

8.47 $$({}^{m-1}|^{\beta}_{m}) = mP_m/2^{m-2} \qquad \text{(by 8.42)}.$$

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8.5. Summarising these properties of $(\mathrm{PA})_m$—

$(^{u-1}\vert_m)\quad (u \leq m-2)$	$(^{m-2}\vert^{\alpha\beta}_m) + (^{m-2}\vert^{\beta\beta}_m)$	$(^{m-1}\vert_m)$	
$\binom{m}{u}\mathrm{P}_m/\mathrm{P}_{m-u}$	$m\mathrm{P}_m + m\mathrm{P}_m/2$	$\mathrm{P}_m + m\mathrm{P}_m/2^{m-2}$	
α_{u-1}	α_{m-2}	α_{m-1}	β_{m-1}

Substituting in **1.38**, we have

$$\mathrm{P}_m\left\{\sum_{u=0}^{m-2}(-1)^{m-u}\frac{\binom{m}{u}}{\mathrm{P}_{m-u}} - m - \frac{m}{2} + 1 + \frac{m}{2^{m-2}}\right\} - 1 = 0,$$

i.e.,

8.51
$$\frac{m}{2^{m-2}} + 1 - \frac{3m}{2} + \sum_{r=2}^{m}(-1)^r\frac{\binom{m}{r}}{\mathrm{P}_r} - \frac{1}{\mathrm{P}_m} = 0.$$

This equation could alternatively have been obtained from either of the semi-reciprocals of $(\mathrm{PA})_m$, the elements of

$$2_{(m-4)\,1}$$

being—

$(^0\vert_m)$	$(^1\vert_m)$	$(^2\vert_m)$	$(^3\vert_m)$	$(^r\vert_m)\quad (r \geq 4)$	
$m\mathrm{P}_m/2^{m-2}$	$m\mathrm{P}_m/2$	$\binom{m}{3}\mathrm{P}_m/2$	$\binom{m}{4}\mathrm{P}_m$	$\binom{m}{r+1}\mathrm{P}_m + \binom{m}{r}\mathrm{P}_m/\mathrm{P}_r$	
		α_2	α_3	α_r	$2_{(r-4)\,1}$

and those of

$$1_{(m-4)\,2}$$ —

$(^0\vert_m)$	$(^1\vert_m)$	$(^2\vert_m)$	$(^3\vert_m)$		$(^r\vert_m)\quad (r \geq 4)$		
P_m	$\binom{m}{3}\mathrm{P}_m/2$	$2\binom{m}{4}\mathrm{P}_m$	$5\binom{m}{5}\mathrm{P}_m/2 + \binom{m}{4}\mathrm{P}_m$		$(r+2)\binom{m}{r+2}\mathrm{P}_m/2 + (r+1)\binom{m}{r+1}\mathrm{P}_m/2^{r-1} + \binom{m}{r}\mathrm{P}_m/\mathrm{P}_r$		
		α_2	$1_{10} = \alpha_3$	$1_{01} = \alpha_3$	α_r	$h\gamma_r$	$1_{(r-4)\,2}$

8.6. In accordance with the principle of **1.51**, we can suppose **8.51** to be true even when $m = 9$, if we put

8.61
$$\mathrm{P}_9 = \infty.$$

The particular cases of **8.51**, along with the fact that we are dealing with positive integers, just suffice to determine the rest of the P's. By **8.47**, $\frac{3}{8}\mathrm{P}_6$, $\frac{7}{32}\mathrm{P}_7$, $\frac{1}{8}\mathrm{P}_8$ are integers; so, if we put

8.62
$$\mathrm{P}_6 = 8x, \quad \mathrm{P}_7 = 32y, \quad \mathrm{P}_8 = 8z,$$

then x, y, z must be integers.

$m = 2$ in 8.51 gives

$$2 + 1 - 3 + (1 - 1)/P_2 = 0 \qquad \text{(identity)}.$$

$m = 3$ and $m = 4$ give

$$\tfrac{3}{2} + 1 - \tfrac{9}{2} + 3/P_2 - 2/P_3 = 0,$$

$$1 + 1 - 6 + 6/P_2 - 4/P_3 = 0,$$

both of which reduce to

$$3/P_2 - 2/P_3 = 2,$$

whence

8.63 $$P_2 = 1$$

and

8.64 $$P_3 = 2.$$

$m = 5$ and $m = 6$ give

$$\tfrac{5}{8} + 1 - \tfrac{15}{2} + 10 - 5 + 5/P_4 - 2/P_5 = 0,$$

$$\tfrac{3}{8} + 1 - 9 + 15 - 10 + 15/P_4 - 6/P_5 = 0,$$

both of which reduce to

$$5/P_4 - 2/P_5 = \tfrac{7}{8},$$

whence

8.65 $$P_4 = 5$$

and

8.66 $$P_5 = 16.$$

(For, since P_4 would be fractional if $P_5 = 1$, we must have $P_5 \geq 2$, so that

$$1 + \tfrac{7}{8} \geq 5/P_4 > \tfrac{7}{8},$$

$$\tfrac{8}{3} \leq P_4 < \tfrac{40}{7},$$

$$P_4 = 3, 4 \text{ or } 5,$$

and correspondingly

$$P_5 = \tfrac{48}{19}, \tfrac{16}{3} \text{ or } 16.)$$

$m = 7$ and $m = 8$ give

$$\tfrac{7}{32} + 1 - \tfrac{21}{2} + 21 - \tfrac{35}{2} + 7 - \tfrac{21}{16} + 7/P_6 - 2/P_7 = 0,$$

$$\tfrac{1}{8} + 1 - 12 + 28 - 28 + 14 - \tfrac{7}{2} + 28/P_6 - 8/P_7 = 0,$$

both of which reduce to

$$7/P_6 - 2/P_7 = \tfrac{3}{32}$$

or

8.67 $$\frac{14}{x} - \frac{1}{y} = \tfrac{3}{2}$$

(in the notation of 8.62).

Finally, $m = 9$ gives

$$\tfrac{9}{128} + \dot{1} - \tfrac{27}{2} + 36 - 42 + \tfrac{126}{5} - \tfrac{63}{8} + 84/P_6 - 36/P_7 + 9/P_8 - 2/P_9 = 0,$$

which, in virtue of 8.61, reduces to

$$28/P_6 - 12/P_7 + 3/P_8 = \tfrac{707}{1920}$$

or

$$\frac{28}{x} - \frac{3}{y} + \frac{3}{z} = \tfrac{707}{240}.$$

But, by 8.67,

$$\frac{28}{x} - \frac{2}{y} = 3.$$

Hence, by subtraction,

8.68 $$\frac{1}{y} - \frac{3}{z} = \tfrac{13}{240}.$$

8.7. We have now to solve the indeterminate equations 8.67 and 8.68. By 8.67, since x would be fractional if $y = 1$,

$$\tfrac{1}{2} + \tfrac{3}{2} \geq \frac{14}{x} > \tfrac{3}{2},$$

$$7 \leq x < \tfrac{28}{3},$$

$$\left.\begin{aligned} x &= 7, \quad 8 \quad \text{or} \quad 9, \\ y &= 2, \quad 4 \quad \text{or} \quad 18, \\ z &= \tfrac{720}{107}, \quad \tfrac{720}{47} \quad \text{or} \quad 2160. \end{aligned}\right\}$$

and correspondingly

so that, by 8.68,

But z must be an integer. Hence the *unique* solution :

8.71 $$x = 9, \qquad y = 18, \qquad z = 2160.$$

It follows (by 8.62) that

8.72 $$P_6 = 72,$$

8.73 $$P_7 = 576,$$

8.74 $$P_8 = 17280\,;$$

and, by 8.31 (thus solving the problem proposed in 7.65),

$$[2\ 2\ 1] = 720,$$
$$[3\ 2\ 1] = 10080,$$
$$[4\ 2\ 1] = 483840.$$

8.8. By 7.91, the order of the group of symmetries of $(\mathrm{PA})_m$ is

8.81 $$g_m = 12\,(m-3)\,!\ [(m-4)\ 2\ 1] = m\,!\ P_m,$$

which takes the following values :—

m	2	3	4	5	6	7	8	9
g_m	2	12	120	1920	51840	2903040 $= 8\times 9\,!$	696729600 $= 192\times 10\,!$	∞

8.9. Let

$$S_m = 2\,[(m-5)\;3\;1]/\binom{m}{4} \quad \text{and} \quad I_m = 3\,[(m-5)\;2\;2]/\binom{m}{4}.$$

Then the numerical properties of $(SA)_m$ can be expressed in the form—

$\binom{u-1}{}\!\big\vert_m$ $(u \leq m-3)$	$\binom{m-3}{}\big\vert\binom{\alpha\alpha\beta}{m} + \binom{m-3}{}\big\vert\binom{\beta\beta\beta}{m}$	$\binom{m-2}{}\big\vert_m$		$\binom{m-1}{}\big\vert_m$	
$\binom{m}{u} S_m/S_{m-u}$	$\binom{m}{2} S_m + \binom{m}{2} S_m/3$	$mS_m + \binom{m}{2} S_m/2^{m-3}$		$S_m + mS_m/P_{m-1}$	
α_{u-1}	α_{m-3}	α_{m-2}	β_{m-2}	α_{m-1}	$(PA)_{m-1}$

(the type-symbols referring to the α_{m-2}'s and β_{m-2}'s which meet at an α_{m-3}), and those of $(IA)_m$—

$\binom{u-1}{}\!\big\vert_m$ $(u \leq m-2)$	$\binom{m-2}{}\big\vert_m$		$\binom{m-1}{}\big\vert_m$
$\binom{m}{u} I_m/I_{m-u}$	$mI_m/2 + \binom{m}{2} I_m/2^{m-3}$		mI_m/P_{m-1}
α_{u-1}	α_{m-2}	β_{m-2}	$(PA)_{m-1}$

Analogously to 8.81, the values of g_m for $(SA)_m$ and $(IA)_m$ are respectively

$$m\,!\;S_m \quad \text{and} \quad m\,!\;I_m.$$

Here are the actual values of S_m and I_m, with those of P_m for comparison :

m	2	3	4	5	6	7	8	9
P_m	1	2	5	16	72	576	17280	∞
S_m		1	2	6	32	576	∞	
I_m		$\frac{4}{3}$	3	12	144	∞		

The explicit expression of

$$P_m, \qquad S_m, \qquad I_m,$$

in terms of m, involves the "SCHLÄFLI functions," on which a paper should appear shortly.

Actually, in the notation of § VII of SCHLÄFLI's "Réduction ..." (*loc. cit.* in Preface),

$$2^m/(m!\ P_m) = f_m(\mu_8, \tfrac{1}{3}\pi, \tfrac{1}{3}\pi, \tfrac{1}{3}\pi, \ldots) + 2f_m(\tfrac{1}{2}\pi - \tfrac{1}{2}\mu_8, \tfrac{1}{4}\pi, \tfrac{1}{3}\pi, \tfrac{1}{3}\pi, \ldots)$$
$$= F_m(\mu_8) + 2G_m(\tfrac{1}{2}\pi - \tfrac{1}{2}\mu_8),$$

$$2^m/(m!\ S_m) = f_m(\mu_7, \tfrac{1}{3}\pi, \tfrac{1}{3}\pi, \tfrac{1}{3}\pi, \ldots) + f_m(\tfrac{2}{3}\pi - \mu_7, \mu_8, \tfrac{1}{3}\pi, \tfrac{1}{3}\pi, \ldots)$$
$$+ 2f_m(\tfrac{1}{3}\pi, \tfrac{1}{2}\pi - \tfrac{1}{2}\mu_8, \tfrac{1}{4}\pi, \tfrac{1}{3}\pi, \ldots),$$

$$2^m/(m!\ I_m) = f_m(2\mu_4, \mu_8, \tfrac{1}{3}\pi, \tfrac{1}{3}\pi, \ldots) + 2f_m(\tfrac{1}{2}\pi - \mu_4, \tfrac{1}{2}\pi - \tfrac{1}{2}\mu_8, \tfrac{1}{4}\pi, \tfrac{1}{3}\pi, \ldots);$$

where $\mu_p = \tfrac{1}{2}\sec^{-1} p$.

9. *Eight-dimensional Co-ordinates.*

9.1. Consider the infinite set of points in eight dimensions whose Cartesian co-ordinates are either all even or all odd and add up to a multiple of 4. These points are the vertices of a degenerate nine-dimensional polytope which we seek to identify with

$$(PA)_9\, 2\sqrt{2}.$$

That the points are equivalent (in the sense of 1.6) may be seen by applying certain symmetries (in this case *translations*) which we call

$$R \quad \text{and} \quad U_{ij}{}^* \qquad (i, j = 1, 2, 3, 4, 5, 6, 7, 8;\ i \neq j).$$

R increases every co-ordinate by 1.

U_{ij} increases x_i and x_j (the ith and jth co-ordinates) each by 2, leaving the remaining six co-ordinates unchanged. (Thus

$$R^2 = U_{12}\,U_{34}\,U_{56}\,U_{78}.)$$

Products of these symmetries clearly suffice to change any point of the set into any other.

9.2. The points nearest to (*i.e.*, distant $2\sqrt{2}$ from) any particular point of the set, are 240 in number. For, those nearest to the origin (0, 0, 0, 0, 0, 0, 0, 0) are

9.21 $\left\{\begin{array}{l} \pm(2, 2, 0, 0, 0, 0, 0, 0), \\ \text{and} \\ (1, 1, 1, 1, 1, 1, 1, 1) \quad \text{with} \quad 0, 2, 4, 6 \text{ or } 8 \text{ minuses.} \end{array}\right.$

We shall eventually identify these 240 points with the vertices of

$$(PA)_8\, 2\sqrt{2}.$$

They possess symmetries which we call

$$S,\ T_{ij} \quad \text{and} \quad (ij) \qquad (i, j = 1, 2, 3, 4, 5, 6, 7, 8;\ i \neq j).$$

* Here, and generally, whenever two or more suffix numbers occur without commas between, they are supposed to be permutable, *e.g.*, $U_{ij} = U_{ji}$.

S diminishes every co-ordinate by the quarter sum of the co-ordinates.

T_{ij} changes the sign of x_i and of x_j, leaving the remaining six co-ordinates unchanged.

(ij) is the "transposition" which interchanges the co-ordinates x_i and x_j, leaving the rest unchanged.

Thus S is the *reflection* in the 7-space

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 0,$$

T_{ij} is the *rotation* (through angle π) about the 6-space

$$x_i = 0 = x_j,$$

and (ij) is the *reflection* in the 7-space

$$x_i = x_j.$$

The points 9.21 are equivalent. For, T_{ij} gives all the necessary changes of sign, (ij) gives all the required permutations, and finally

$$(2, 2;\ 0, 0, 0, 0, 0, 0)\ T_{12}\, ST_{12} = (1, 1, 1, 1, 1, 1, 1, 1).$$

Actually, (ij) is expressible in terms of S and T_{ij}. For

9.22 $$(ij) = (ST_{ij})^3 \quad \text{or} \quad (T_{ij}S)^3.$$

The following are the simplest properties of S and T_{ij}:

$$S^2 = 1,$$

$$(ST_{ij})^6 = 1,$$

9.23 $$T_{ij}T_{kl} = T_{li}T_{kj}.$$

The convention

$$T_{kk} = 1$$

makes 9.23 include

$$T_{ij} = T_{ji} = T_{ki}\,T_{kj}, \quad T_{ij}\,T_{kl} = T_{kl}\,T_{ij}, \quad T_{ij}{}^2 = 1,$$

all of which are trivial.

S and T_{ij}, being symmetries also of the original infinite set of points, are related to R and U_{ij} by the equations

$$(RS)^2 = 1 = (T_{ij}\,U_{ij})^2,$$

$$R = SU_{ij}^{-1}\,SU_{ij} \quad = ST_{ij}\,SU_{ij}\,ST_{ij}\,S \quad = T_{ij}\,SU_{ij}^{-1}\,ST_{ij},$$

$$U_{ij} = T_{ij}\,R^{-1}\,T_{ij}\,R = T_{ij}\,ST_{ij}\,RT_{ij}\,ST_{ij} = ST_{ij}\,R^{-1}\,T_{ij}\,S.$$

Note that these relations remain true if

$$R, \quad S, \quad T_{ij}, \quad U_{ij}$$

are replaced respectively by

$$U_{ij}, \quad T_{ij}, \quad S, \quad R.$$

It will be found convenient to let

$$T_{ijkl} = T_{ij}\,T_{kl}$$

and

$$T = T_{1234}\,T_{5678} \quad (\textit{i.e.}, \text{ reflection in the origin}),$$

so that $(ST_{ijkl})^4 = 1 = (ST)^2$. Note that

$$(1, 1, 1, 1, 1, 1, 1, 1)\,ST = (1, 1, 1, 1, 1, 1, 1, 1).$$

9.3. Of the 240 points 9.21, those nearest to (*i.e.*, distant $2\sqrt{2}$ from) any one, are 56 in number. For, those nearest to (1, 1, 1, 1, 1, 1, 1, 1) are

9.31 $$(2, 2, 0, 0, 0, 0, 0, 0) \quad \text{and} \quad (-1, -1, 1, 1, 1, 1, 1, 1).$$

For simplicity, let

9.32 $$\begin{cases} C_{12} = (2, 2\,;\; 0, 0, 0, 0, 0, 0), \\ c_{12} = (-1, -1\,;\; 1, 1, 1, 1, 1, 1). \end{cases}$$

Then the 56 points 9.31 are simply

9.33 $$C_{ij} \quad \text{and} \quad c_{ij},$$

where i and j can be any unequal pair of the numbers 1, 2, 3, 4, 5, 6, 7, 8.

These points, lying in the 7-space

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 4,$$

are to be identified with the vertices of

$$(PA)_7\, 2\sqrt{2}.$$

They are equivalent, since the transformation $2R^{-1}$ puts them into the symmetrical form

$$(3, 3, -1, -1, -1, -1, -1, -1), \;(-3, -3, 1, 1, 1, 1, 1, 1).$$

Besides the obvious symmetries—(ij), which interchanges the suffix-numbers i and j wherever they occur; and ST, which interchanges C and c, leaving the suffixes unchanged—the points 9.31 or 9.33 possess also the symmetry $T_{ijkl}\,ST_{ijkl}$ (i, j, k, l being all different). This is the reflection in

$$x_e + x_f + x_g + x_h = x_i + x_j + x_k + x_l,$$

where e, f, g, h are the rest of the numbers 1, 2, 3, 4, 5, 6, 7, 8. Thus

$$T_{ijkl}\,ST_{ijkl} = T_{efgh}\,ST_{efgh}.$$

Introducing a new notation, let

9.34 $$[efgh\,.\,ijkl] = T_{ijkl}\,ST_{ijkl}.$$

Naturally

$$[efgh\,.\,ijkl] = [ijkl\,.\,efgh],$$

and the order of the numbers on one side of the dot is quite arbitrary. This new symmetry interchanges C_{ef}, c_{gh}; C_{ij}, c_{kl}; and so on; but leaves, *e.g.*, C_{el} and c_{el} unchanged. It is called a "bifid reflection," by analogy with CAYLEY's "bifid substitution."*

Note that

9.35 $$(ij) = [efgi\,.\,jhkl]\,[efgj\,.\,ihkl]\,[efgi\,.\,jhkl]$$

and

9.36 $$ST = [3567\,.\,1248]\,[1467\,.\,2358]\,[1257\,.\,3468]\,[1236\,.\,4578]\,[2347\,.\,1568]\,[1345\,.\,2678]\,[2456\,.\,1378].$$

(The order of these seven factors is quite immaterial. The essential thing is that every pair have just two common numbers on each side of the dot.)

It is convenient to omit the numbers 7 and 8 (in a bifid reflection) when they occur respectively *before* and *after* the dot. Thus we write

9.37 $$[fgh\,.\,ijk] = [fgh7\,.\,ijk8].$$

Of course

$$[fgh\,.\,ijk] \neq [ijk\,.\,fgh].$$

9.4. Of the 56 points 9.33, those nearest to (*i.e.*, distant $2\sqrt{2}$ from) any one, are 27 in number. For, those nearest to C_{78} are

9.41 $$(2, 0, 0, 0, 0, 0\,;\; 2, 0) \quad \text{and} \quad (-1, \;\; 1, 1, 1, 1, 1,\,;\; 1, 1).$$

Changing the notation by putting

9.42 $$a_i = C_{i7} \quad \text{and} \quad b_i = C_{i8} \quad (i = 1, 2, 3, 4, 5, 6),$$

these 27 points are simply

9.43 $$a_i, \quad b_i \quad \text{and} \quad c_{ij},$$

where i and j can be any unequal pair of the numbers 1, 2, 3, 4, 5, 6.

These points, lying in the 6-space

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 2 = x_7 + x_8,$$

will shortly be identified with the vertices of

$$(PA)_6\, 2\sqrt{2}.$$

* SALMON's "Higher Plane Curves," § 261.

That they are equivalent, may be seen by considering the 20 symmetries $[fgh \,.\, ijk]$, where f, g, h, i, j, k are all the numbers 1, 2, 3, 4, 5, 6, arranged in any order. Expressed in terms of transpositions of the symbols,

9.44 $$[fgh \,.\, ijk] = (a_f\, c_{gh})\,(a_g\, c_{fh})\,(a_h\, c_{fg})\,(b_i\, c_{jk})\,(b_j\, c_{ik})\,(b_k\, c_{ij}).$$

Of the 27 points 9.43, those nearest to (*i.e.*, distant $2\sqrt{2}$ from) any one, are 16 in number. For, those nearest to a_6 are

9.45 $$b_6, \quad c_{ij} \quad \text{and} \quad a_i \quad (i, j = 1, 2, 3, 4, 5\,;\; i \neq j).$$

Applying the congruent transformation $T_{68}SU_{68}TR$*, these 16 points become respectively

$$(0, 0, 0, 0, 0\,;\; 0, 0, 0), \quad (2, 2, 0, 0, 0\,;\; 0, 0, 0) \quad \text{and} \quad (0, 2, 2, 2, 2\,;\; 0, 0, 0).$$

By 6.22, they are the vertices of

$$h\gamma_5\, 2\sqrt{2}.$$

9.5. Since (by 8.14)

$$h\gamma_5 = (\mathrm{PA})_5,$$

and since (by 8.2) $(\mathrm{PA})_m$ is always the vertex figure of $(\mathrm{PA})_{m+1}$ (when the latter exists), the rule 6.1 enables us successively to identify the sets of points

$$9.43, \quad 9.33, \quad 9.21 \quad \text{and} \quad 9.1,$$

with the vertices of

$$(\mathrm{PA})_6\, 2\sqrt{2}, \quad (\mathrm{PA})_7\, 2\sqrt{2}, \quad (\mathrm{PA})_8\, 2\sqrt{2} \quad \text{and} \quad (\mathrm{PA})_9\, 2\sqrt{2},$$

respectively. Conditions (i) and (ii) (of 6.1) are clearly satisfied in each case. Condition (iii) is automatically satisfied for such bounding figures as are α's, since then no " other points of the original set " are required. So we have only to consider the β bounding figures.

Now, if we are given one vertex of β_m and the vertices of the actual vertex figure at this vertex, there remains only *one* more vertex of β_m, this vertex being the image of the first vertex in the centre. Also the centre of β_m is the centre of the vertex figure. Thus if, in (iii), the " typical bounding figure " of the " $(m-1)$-dimensional polytope " is a $\beta_{m-2}\times$whose centre is O, then we have only to show that the image of A in O belongs to the given set of points.

Taking $$b_6,\, c_{12},\, c_{13},\, c_{23},\, c_{14},\, c_{24},\, c_{34},\, a_5$$

as the vertices of a typical bounding $\beta_4 2\sqrt{2}$ of the $h\gamma_5\, 2\sqrt{2}$ 9.45, O is

$$(0, 0, 0, 0\,;\; 1, 1, 1, 1),$$

* Products of operations are, in this paper, to be worked from left to right. Thus, in the present case, we apply T_{68} first and R last.

and the successive A's and images are as follows :—

Polytope to be identified	$(PA)_6\ 2\sqrt{2}$	$(PA)_7\ 2\sqrt{2}$	$(PA)_8\ 2\sqrt{2}$	$(PA)_9\ 2\sqrt{2}$
A	a_6	C_{78}	(1, 1, 1, 1, 1, 1, 1, 1)	(0, 0, 0, 0, 0, 0, 0, 0)
Image of A in O	b_5	C_{56}	(−1, −1, −1, −1; 1, 1, 1, 1)	(0, 0, 0, 0; 2, 2, 2, 2)

Thus the identification is complete.

Incidentally, we observe that the numbers

$$16, \quad 27, \quad 56, \quad 240, \quad \infty,$$

agree with the formula 8.33.

We have now established the existence of

$$(PA)_m \quad (\text{for } m \leq 9),$$

i.e., of

$$n_{21} \quad (\text{for } n \leq 5).$$

The existence of

$$2_{n1} \quad \text{and} \quad 1_{n2}$$

follows by semi-reciprocation (7.8). Of the fourteen polytopes 7.45, we have thus established all save three, namely

$$(SA)_8 = 3_{31},$$

$$1_{33} \quad (\text{its semi-reciprocal})$$

and

$$(IA)_7 = 2_{22}.$$

9.6. Consider now the totality of points whose eight co-ordinates, all even or all odd, add up to *zero*. These points, whose equivalence can be established by means of the symmetries

$$T_{ijkl}\, RT_{ijkl},$$

are the vertices of a degenerate eight-dimensional polytope which can be regarded as the section of $(PA)_9\ 2\sqrt{2}$ by the 7-space

9.61 $$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 0.$$

This polytope will be found to be

$$(SA)_8\ 2\sqrt{2}.$$

The 126 points distant $2\sqrt{2}$ from

9.62 $$(0, 0, 0, 0, 0, 0, 0, 0)$$

are

9.63 $$(2, 0, 0, 0, 0, 0, 0, -2) \quad \text{and} \quad (1, 1, 1, 1, -1, -1, -1, -1).$$

These points are equivalent by means of the bifid reflections $T_{ijkl}!ST_{ijkl}$.
They will be identified with the vertices of

$$(SA)_7\ 2\sqrt{2}.$$

Of these, the 32 points distant $2\sqrt{2}$ from

9.64 $$(2\,;\ 0, 0, 0, 0, 0, 0\,;\ -2)$$

are

$$(0\,;\ 2, 0, 0, 0, 0, 0\,;\ -2),\quad (1\,;\ 1, 1, 1, -1, -1, -1\,;\ -1),\quad (2\,;\ 0, 0, 0, 0, 0, -2\,;\ 0).$$

By means of the transformation $T_{18}SR$, these points become recognisable as the vertices

9.65 $$(0\,;\ 2, 0, 0, 0, 0, 0\,;\ 2),\quad (0\,;\ 2, 2, 2, 0, 0, 0\,;\ 2),\quad (0\,;\ 2, 2, 2, 2, 2, 0\,;\ 2)$$

of $h\gamma_6\ 2\sqrt{2}$ (6.23).

Now, by 8.15,

$$h\gamma_6 = (SA)_6\,;$$

and (by 8.9) $(SA)_m$ is bounded by α_{m-1}'s and $(PA)_{m-1}$'s. For the purposes of 6.1, we need only consider the $(PA)_{m-1}$'s. A typical bounding $(PA)_5\ 2\sqrt{2}$ or $h\gamma_5\ 2\sqrt{2}$ of the $h\gamma_6\ 2\sqrt{2}$ 9.65, has the vertices

$$(0\,;\ 2, 0, 0, 0, 0\,;\ 0\,;\ 2),\quad (0\,;\ 2, 2, 2, 0, 0\,;\ 0\,;\ 2),\quad (0\,;\ 2, 2, 2, 2, 2\,;\ 0\,;\ 2),$$

which, by the reverse transformation $R^{-1}ST_{18}$, become

$$(0\,;\ 2, 0, 0, 0, 0\,;\ 0\,;\ -2),\ (1\,;\ 1, 1, 1, -1, -1\,;\ -1\,;\ -1),\ (2\,;\ 0, 0, 0, 0, 0\,;\ -2\,;\ 0).$$

These points, along with 9.64 and certain other points from 9.63, make up the complete sets of vertices

9.66 $$(2, 0, 0, 0, 0, 0\,;\ 0, -2)\quad \text{and}\quad (1, 1, 1, 1, -1, -1\,;\ -1, -1)$$

of a $(PA)_6\ 2\sqrt{2}$ (obtainable from 9.41 by means of the transformation U_{78}^{-1}). Hence the points 9.63 are the vertices of $(SA)_7\ 2\sqrt{2}$, of which a typical bounding $(PA)_6\ 2\sqrt{2}$ has the vertices 9.66.

But the points 9.66, along with 9.62 and certain other points of the infinite set 9.6, make up the complete set of vertices of that $(PA)_7\ 2\sqrt{2}$ which is obtained from 9.31 by means of the transformation U_{78}^{-1}. Hence the points 9.6 are the vertices of $(SA)_8\ 2\sqrt{2}$.

The existence of

$$(SA)_8 \quad \text{or} \quad 3_{31}$$

is thus established. That of

$$1_{33}$$

follows by semi-reciprocation. There remains now only 2_{22}.

9.7. Since $T_{78}ST_{78}$, a symmetry of $(PA)_9\, 2\sqrt{2}$, transforms 9.61 into $x_7+x_8=0$, we have the interesting fact that the section of $(PA)_9\, 2\sqrt{2}$ by the 7-space $x_7+x_8=0$ is another $(SA)_8\, 2\sqrt{2}$.

9.8. The common part of these two $(SA)_8\, 2\sqrt{2}$'s, *i.e.*, the section of $(PA)_9\, 2\sqrt{2}$ by the 6-space

9.81 $$x_1+x_2+x_3+x_4+x_5+x_6=0=x_7+x_8,$$

is a degenerate seven-dimensional polytope, which we shall identify with

$$(IA)_7\, 2\sqrt{2}.$$

Of its vertices, which are equivalent by means of the symmetries

$$T_{ijk8}\, RT_{ijk8} \quad (i, j, k, = 1, 2, 3, 4, 5, 6\, ;\ i \neq j \neq k \neq i),$$

those distant $2\sqrt{2}$ from

9.82 $$(0, 0, 0, 0, 0, 0, 0, 0)$$

are

9.83 $$(0, 0, 0, 0, 0, 0\, ;\ 2, -2), \quad (1, 1, 1, -1, -1, -1\, ;\ 1, -1),$$
$$(2, 0, 0, 0, 0, -2\, ;\ 0, 0).$$

These 72 points, which will soon be seen to be the vertices of

$$(IA)_6\, 2\sqrt{2},$$

are equivalent by means of the bifid reflections

$$T_{ijk8}\, ST_{ijk8} \quad (i, j, k = 1, 2, 3, 4, 5, 6\, ;\ i \neq j \neq k \neq i).$$

(Note that they, unlike 9.41, possess also the symmetry T.)

Of these points, those distant $2\sqrt{2}$ from

9.84 $$(0, 0, 0, 0, 0, 0\, ;\ -2\, ;\ 2)$$

are

9.85 $$(1, 1, 1, -1, -1, -1\, ;\ -1\, ;\ 1).$$

The transformation R makes these 20 points recognisable as the vertices of

$$t_2\alpha_5\, 2\sqrt{2}.$$

Now, by 8.16,

$$t_2\alpha_5 = (IA)_5\, ;$$

and (by 8.9) $(IA)_m$ is bounded entirely by $(PA)_{m-1}$'s, these being all equivalent. A typical bounding $(PA)_4\, 2\sqrt{2}$ or $t_1\alpha_4\, 2\sqrt{2}$ of the $t_2\alpha_5\, 2\sqrt{2}$ 9.85, has the vertices

$$(1, 1, 1, -1, -1\, ;\ -1\, ;\ -1\, ;\ 1).$$

These points, along with 9.84 and

$$(2,\ 0,\ 0,\ 0,\ 0\ ;\ \ -2\ ;\ \ 0\ ;\ \ 0)$$

(which also occur among 9.83), make up the complete set of vertices of a $(\mathrm{PA})_5\, 2\sqrt{2}$ or $h\gamma_5\, 2\sqrt{2}$ (obtainable from 9.45 by means of the transformation U_{67}^{-1}). Hence the points 9.83 are the vertices of $(\mathrm{IA})_6\, 2\sqrt{2}$, of which this $(\mathrm{PA})_5\, 2\sqrt{2}$ is a typical bounding figure.

But the vertices of this typical bounding figure, along with 9.82 and certain other points satisfying 9.81, make up the complete set of vertices of the $(\mathrm{PA})_6\, 2\sqrt{2}$ which we obtain from 9.41 by means of the transformation U_{67}^{-1}. Hence the infinite set of points whose eight co-ordinates, all even or all odd, satisfy 9.81, are the vertices of $(\mathrm{IA})_7\, 2\sqrt{2}$.

Thus we have established the existence of

$$(\mathrm{IA})_7 \quad \text{or} \quad 2_{22},$$

the last of the polytopes 7.45.

9.9. By applying certain other symmetries of $(\mathrm{PA})_8\, 2\sqrt{2}$ in the way in which $\mathrm{T}_{78}\mathrm{ST}_{78}$ was applied in 9.7, it is found that there are in all 120 sections of $(\mathrm{PA})_9\, 2\sqrt{2}$, through any one vertex, which are $(\mathrm{SA})_8\, 2\sqrt{2}$'s ; namely, one section for every pair of opposite vertices of the vertex figure, $(\mathrm{PA})_8$. The 7-spaces of these sections are as follows :—

$$\begin{cases} (1) & x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8=0, \\ 28 \text{ like} & x_1+x_2+x_3+x_4+x_5+x_6-x_7-x_8=0, \\ 35 \text{ like} & x_1+x_2+x_3+x_4-x_5-x_6-x_7-x_8=0, \\ 28 \text{ like} & x_7-x_8=0, \\ 28 \text{ like} & x_7+x_8=0. \end{cases}$$

These 7-spaces may be called " primes of symmetry " of $(\mathrm{PA})_9\, 2\sqrt{2}$ or of $(\mathrm{PA})_8\, 2\sqrt{2}$. For, the reflections in them, *viz.*,

$$\mathrm{S},\quad \mathrm{T}_{ij}\mathrm{ST}_{ij},\ \mathrm{T}_{ijkl}\mathrm{ST}_{ijkl},\quad (\mathrm{ST}_{ij})^3 \text{ or } (ij),\quad (\mathrm{ST}_{ij})^2\mathrm{S} \text{ or } (ij)\ \mathrm{T}_{ij},$$

are symmetries of the polytopes.

10. *Nine-dimensional Co-ordinates.*

10.1. Consider the infinite set of points whose *nine* Cartesian co-ordinates are mutually congruent modulo 3 and add up to zero. These points, lying in the 8-space

10.11 $$x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8+x_9=0,$$

can be identified with the vertices of

$$(\mathrm{PA})_9\, 3\sqrt{2}$$

by applying to them the transformation

$$\tfrac{2}{3}\,\Omega T_8,$$

where

10.12 $$\Omega = \tfrac{1}{6}\begin{pmatrix} 5 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 \\ -1 & 5 & -1 & -1 & -1 & -1 & -1 & -1 & 2 \\ -1 & -1 & 5 & -1 & -1 & -1 & -1 & -1 & 2 \\ -1 & -1 & -1 & 5 & -1 & -1 & -1 & -1 & 2 \\ -1 & -1 & -1 & -1 & 5 & -1 & -1 & -1 & 2 \\ -1 & -1 & -1 & -1 & -1 & 5 & -1 & -1 & 2 \\ -1 & -1 & -1 & -1 & -1 & -1 & 5 & -1 & 2 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & 5 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix}$$

and T_8 changes the sign of x_8. (It is easily verified that Ω satisfies the conditions which make it a *congruent* transformation.)

Since we are only considering points satisfying 10.11, the relation

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9)\,\tfrac{2}{3}\Omega T_8 = (x'_1, x'_2, x'_3, x'_4, x'_5, x'_6, x'_7, x'_8, x'_9)$$

implies

10.13 $$\begin{cases} x'_r = \tfrac{1}{3}\,(2x_r + x_9) & (r < 8), \\ x'_8 = -\,\tfrac{1}{3}\,(2x_8 + x_9), \\ x'_9 = 0. \end{cases}$$

The general point

10.14 $$(3y_1 + z,\quad 3y_2 + z,\quad 3y_3 + z,\quad 3y_4 + z,\quad 3y_5 + z,\quad 3y_6 + z,\quad 3y_7 + z,\quad 3y_8 + z,\quad 3y_9 + z)$$

of the set considered, therefore becomes

10.15 $$(2y_1 + z',\quad 2y_2 + z',\quad 2y_3 + z',\quad 2y_4 + z',\quad 2y_5 + z',\quad 2y_6 + z',\quad 2y_7 + z',\quad 2y'_8 + z',\quad 0),$$

where

$$\begin{cases} z' = y_9 + z, \\ y'_8 = -\,y_8 - y_9 - z. \end{cases}$$

The sum of the new co-ordinates is

$$\sum_{r=1}^{9} x'_r = \tfrac{2}{3}\sum_{r=1}^{9} x_r + \tfrac{4}{3}\,(x_9 - x_8) = \tfrac{4}{3}\,(x_9 - x_8) = 4\,(y_9 - y_8).$$

Thus we have obtained a vertex of $(\mathrm{PA})_9\; 2\sqrt{2}$ as given in 9.1.

Conversely, if the point 10.15, satisfying

$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y'_8 = 2t$$

(which makes the co-ordinates add up to a multiple of 4), is a general vertex of

$$(\mathrm{PA})_9\, 2\sqrt{2},$$

we can make it correspond to the point 10.14 of the original set (10.1) by putting

$$\begin{cases} z = y'_8 - t, \\ y_8 = -y'_8 - z', \\ y_9 = -y'_8 + z' + t. \end{cases}$$

The identification is now complete.

10.2. By taking those points of the set 10.1 which are distant $3\sqrt{2}$ from the origin, we obtain the vertices of

$$(\mathrm{PA})_8\, 3\sqrt{2}$$

in the beautiful form

10.21 (2, 2, 2, − 1, − 1, − 1, − 1, − 1, − 1), (3, 0, 0, 0, 0, 0, 0, 0, − 3),
(1, 1, 1, 1, 1, 1, − 2, − 2, − 2).

Thence we obtain two different sets of co-ordinates for the vertices of

$$(\mathrm{PA})_7\, 3\sqrt{2},$$

namely

(2, − 1, − 1, − 1, − 1, − 1 ; 2, 2, − 1), (1, 1, 1, − 2, − 2, − 2 ; 1, 1, 1),
(0, 0, 0, 0, 0, − 3 ; 3, 0, 0)

and

10.22 (0, 0, 0, 0, 0, 0, − 3 ; 0 ; 3), (2, 2, − 1, − 1, − 1, − 1, − 1 ; − 1 ; 2),
(1, 1, 1, 1, 1, − 2, − 2 ; − 2 ; 1), (3, 0, 0, 0, 0, 0, 0 ; − 3 ; 0).

Trivially transforming* the former set to make it more symmetrical, we have, for the vertices of

$$(\mathrm{PA})_7\, 6\sqrt{2},$$

10.23 (5, −1, −1, −1, −1, −1 ; 2, 2, −4), (3, 3, 3, −3, −3, −3 ; 0, 0, 0),
(1, 1, 1, 1, 1, −5 ; 4, −2, −2).

Proceeding one stage further, we get the vertices of

$$(\mathrm{PA})_6\, 3\sqrt{2}$$

in the alternative forms

10.24 (1, 1, 1, 1, 1, −2 ; −2, −2 ; 1), (2, 2, −1, −1, −1, −1 ; −1, −1 ; 2),
(0, 0, 0, 0, 0, −3 ; 0, 0, ; 3)

and

10.25 (0, 0, 0, ; 2, −1, −1 ; 1, 1, −2), (1, 1, −2 ; 0, 0, 0 ; 2, −1, −1),
(2, −1, −1 ; 1, 1, −2 ; 0, 0, 0).

* By means of $2\mathrm{U}_{789}^{-1}$, in the notation of 10.6.

The latter form (10.25) is of special interest, as it corresponds to Mr. P. HALL's notation for the twenty-seven lines on the general cubic surface. For, if we associate the nine co-ordinates with the symbols

$$s_1, s_2, s_3, \quad t_1, t_2, t_3, \quad u_1, u_2, u_3,$$

and accordingly write (*e.g.*),

10.26
$$\begin{cases} t_1 u_3 = (0, 0, 0\,;\ 2\,;\ -1, -1\,;\ 1, 1\,;\ -2), \\ u_1 s_3 = (1, 1\,;\ -2\,;\ 0, 0, 0\,;\ 2\,;\ -1, -1), \\ s_1 t_3 = (2\,;\ -1, -1\,;\ 1, 1\,;\ -2\,;\ 0, 0, 0), \end{cases}$$

then the 27 points 10.25 are represented by the symbols

10.27
$$t_j u_k, \quad u_k s_i, \quad s_i t_j$$

(where $i, j, k = 1, 2, 3$ independently), in such a way that any two of the points are mutually distant 6 or $3\sqrt{2}$ according as the number of the letters s, t, u which occur with different suffixes in the two symbols is even or odd. For instance,

$$s_2t_2,\ s_2t_3,\ s_3t_2,\ s_3t_3, \quad u_1s_1,\ u_2s_1,\ u_3s_1, \quad t_1u_1,\ t_1u_2,\ t_1u_3$$

are all distant 6 from s_1t_1; while the three points specified in 10.26 form a triangle of sides $3\sqrt{2}$.

10.3. In 9.9, we saw that $(\mathrm{SA})_8$ can be obtained as the section of $(\mathrm{PA})_9$ by the 7-space through any vertex perpendicular to any diameter (*i.e.*, join of a pair of opposite vertices) of the vertex figure at that vertex. Of such 7-spaces, 120 pass through the origin. According to the co-ordinates 10.21, these consist of

$$\begin{cases} 84 \text{ like } x_1 + x_2 + x_3 = 0, \\ 36 \text{ like } x_1 - x_2 = 0 \end{cases}$$

(10.11 being understood). In this manner, we obtain the co-ordinates of

$$(\mathrm{SA})_8\ 3\sqrt{2}$$

in various ways as a section of $(\mathrm{PA})_9\ 3\sqrt{2}$.

In 9.8, we saw that the section of $(\mathrm{PA})_9\ 2\sqrt{2}$ by the 6-space 9.81 is $(\mathrm{IA})_7\ 2\sqrt{2}$. It follows (by applying T_{4567}, in the notation of 9.2) that the section by the 6-space

$$x'_1 + x'_2 + x'_3 = x'_4 + x'_5 + x'_6, \qquad x'_7 = x'_8$$

is another $(IA)_7\ 2\sqrt{2}$. Hence, by 10.13, the section of $(PA)_9\ 3\sqrt{2}$ by the 6-space

10.31 $$x_1 + x_2 + x_3 = x_4 + x_5 + x_6 = x_7 + x_8 + x_9 = 0$$

is

$$(IA)_7\ 3\sqrt{2}.$$

Taking points distant $3\sqrt{2}$ from the origin, we obtain the vertices of

$$(IA)_6\ 3\sqrt{2}$$

in the form

10.32 $$\begin{cases} (3, 0, -3;\ 0, 0, 0;\ 0, 0, 0), \quad (0, 0, 0;\ 3, 0, -3;\ 0, 0, 0), \\ \qquad (0, 0, 0;\ 0, 0, 0;\ 3, 0, -3), \\ (2, -1, -1;\ 2, -1, -1;\ 2, -1, -1), \quad (1, 1, -2;\ 1, 1, -2;\ 1, 1, -2). \end{cases}$$

10.4. In 10.2 we considered those vertices of

$$(PA)_9\ 3\sqrt{2}$$

which are at distance $3\sqrt{2}$ from the origin. Let us now consider those which are at a few greater distances.

Distant 6 from the origin, there are 2160 points—

10.41 $$\begin{cases} (5, 2, -1, -1, -1, -1, -1, -1, -1), \quad (1, 1, 1, 1, 1, 1, 1, -2, -5), \\ (4, 1, 1, 1, 1, -2, -2, -2, -2), \quad (3, 3, 0, 0, 0, 0, 0, -3, -3), \\ \qquad (2, 2, 2, 2, -1, -1, -1, -1, -4). \end{cases}$$

Distant $3\sqrt{6}$, there are 6720—

10.42 $$\begin{cases} (6, 0, 0, 0, 0, 0, 0, -3, -3), \quad (3, 3, 0, 0, 0, 0, 0, 0, -6), \\ (5, 2, 2, -1, -1, -1, -1, -1, -4), \quad (4, 1, 1, 1, 1, 1, -2, -2, -5), \\ (4, 4, 1, 1, -2, -2, -2, -2, -2), \quad (2, 2, 2, 2, 2, -1, -1, -4, -4), \\ \qquad (3, 3, 3, 0, 0, 0, -3, -3, -3). \end{cases}$$

Distant $6\sqrt{2}$, there are 17280 + 240—

10.43 $$\begin{cases} (8, -1, -1, -1, -1, -1, -1, -1, -1), \quad (1, 1, 1, 1, 1, 1, 1, 1, -8), \\ (7, 1, 1, 1, -2, -2, -2, -2, -2), \quad (2, 2, 2, 2, 2, -1, -1, -1, -7), \\ (6, 3, 0, 0, 0, 0, -3, -3, -3), \quad (3, 3, 3, 0, 0, 0, 0, -3, -6), \\ (5, 5, -1, -1, -1, -1, -1, -1, -4), \quad (4, 1, 1, 1, 1, 1, 1, -5, -5), \\ (5, 2, 2, 2, -1, -1, -1, -4, -4), \quad (4, 4, 1, 1, 1, -2, -2, -2, -5), \\ \qquad (3, 3, 3, 3, 0, -3, -3, -3, -3), \end{cases}$$

and

$$(4,\ 4,\ 4,\ -2,\ -2,\ -2,\ -2,\ -2,\ -2), \quad (6,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ -6),$$

$$(2,\ 2,\ 2,\ 2,\ 2,\ 2,\ -4,\ -4,\ -4).$$

The last 240 points are obviously the vertices of $(\mathrm{PA})_8\ 6\sqrt{2}$, being the same as 10.21, only doubled throughout.

It is easily proved that the points

10.41, 10.42, 10.43

are respectively the vertices of

$$2_{41}\ 3\sqrt{2}, \quad t_1 4_{21}\ 3\sqrt{2}, \quad 1_{42}\ 3\sqrt{2}.$$

Since these polytopes all have precisely the same symmetries as $(\mathrm{PA})_8$, it is only necessary to identify one vertex of each.

By 7.8, the centres of the 4_{11}'s (β_7's) of 4_{21} ($= (\mathrm{PA})_8$) are the vertices of $2_{14}\times$ ($= 2_{41}\times$). A typical $\beta_7\ 3\sqrt{2}$ of $(\mathrm{PA})_8\ 3\sqrt{2}$ (as given in 10.21) has the vertices

$$(3,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\ ;\ 0\ ;\ -3), \quad (1,\ 1,\ 1,\ 1,\ 1,\ 1,\ -2\ ;\ -2\ ;\ -2).$$

Its centre,

$$(1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1\ ;\ -2\ ;\ -5)\ \tfrac{1}{2},$$

after multiplication by 2, occurs in 10.41.

Similarly, the centres of the 4_{20}'s (α_7's) of 4_{21} are the vertices of $1_{42}\times$.

A typical $\alpha_7\ 3\sqrt{2}$ of $(\mathrm{PA})_8\ 3\sqrt{2}$ has the vertices

$$(3,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\ ;\ -3).$$

Its centre,

$$(1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1\ ;\ -8)\ \tfrac{3}{8},$$

after multiplication by $\tfrac{8}{3}$, occurs in 10.43.

Finally, by the definition of truncation, the centres of the edges of $(\mathrm{PA})_8$ are the vertices of

$$t_1\ (\mathrm{PA})_8\ \times.$$

A typical edge of $(\mathrm{PA})_8\ 3\sqrt{2}$ is terminated by the points

$$(3,\ 0\ ;\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\ ;\ -3).$$

Its centre,

$$(1,\ 1\ ;\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\ ;\ -2)\ \tfrac{3}{2},$$

after multiplication by 2, occurs in 10.42.

10.5. The following table exhibits some particularly interesting sections of $(\mathrm{PA})_9\ 2\sqrt{2}$ and of $(\mathrm{PA})_9\ 3\sqrt{2}$:—

	$(\mathrm{PA})_9\ 2\sqrt{2}$	$(\mathrm{PA})_9\ 3\sqrt{2}$
$x_1 = x_2$	$(\mathrm{SA})_8\ 2\sqrt{2}$	$(\mathrm{SA})_8\ 3\sqrt{2}$
$x_1 = x_2 = x_3$	$(\mathrm{IA})_7\ 2\sqrt{2}$	$(\mathrm{IA})_7\ 3\sqrt{2}$
$x_1 = x_2 = x_3 = x_4$	$h\delta_6\ 2\sqrt{2}$	$h\delta_6\ 3\sqrt{2}$
$x_1 = x_2 = x_3 = x_4 = x_5$	$\alpha_4 h\ 2\sqrt{2}$	$\alpha_4 h\ 3\sqrt{2}$
$x_1 = x_2 = x_3 = x_4 = x_5 = x_6$	$[a_2 h,\ \delta_2]\ 2\sqrt{2}$	$[\alpha_2 h,\ \delta_2]\ 3\sqrt{2}$
$x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = x_7$	$(\alpha_1 \overline{\ {\scriptstyle\sqrt{2}}\ } \alpha_0)\ h\ 2\sqrt{2}$	$(\alpha_1 \overline{\ {\scriptstyle\sqrt{2}}\ } \alpha_0)\ h\ 3\sqrt{2}$
$x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = x_7 = x_8$	$\delta_2\ 2\sqrt{2}$	$\delta_2\ 6\sqrt{2}$

Here

$$(\alpha_1 \overline{\ {\scriptstyle\sqrt{2}}\ } \alpha_0)\ h$$

stands for an infinity of $(\mathrm{PA})_2$'s (isosceles triangles of sides 1, $\sqrt{2}$, $\sqrt{2}$) filling a plane. It can be obtained by uniformly compressing a {3, 6} or $\alpha_2 h$ in the direction of one edge.

The fact that corresponding sections of $(\mathrm{PA})_9\ 2\sqrt{2}$ and of $(\mathrm{PA})_9\ 3\sqrt{2}$ are similar, in the linear ratio 2 : 3, except in the last case (eight co-ordinates equal), is merely a consequence of 10.13.

10.6. $(\mathrm{PA})_9\ 3\sqrt{2}$ clearly possesses 84 symmetries U_{ijk} and 84 symmetries V_{ijk}, defined as follows :—

U_{ijk} increases the co-ordinates x_i, x_j, x_k each by 2, and diminishes the remaining six each by 1.

V_{ijk} diminishes x_i, x_j, x_k each by two-thirds of their sum, and diminishes each of the remaining six co-ordinates by one-third of the sum of those six.

Thus U_{ijk} is a translation (through distance $3\sqrt{2}$) ; and V_{ijk} is the rotation (through angle π) about, or the reflection in, the 7-space

$$x_i + x_j + x_k = 0 = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9,$$

according as we are considering the whole 9-space or only the 8-space in which $(\mathrm{PA})_9\ 3\sqrt{2}$ lies.

Let $c, d, e, f, g, h, i, j, k$ denote all the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, arranged in arbitrary order, and let (ij) (as usual) denote the transposition of the two co-ordinates x_i, x_j. The simplest relations between our new symmetries are as follows :—

10.61 $$V_{ijk}^2 = (V_{ghk} V_{ijk})^2 = (U_{ijk} V_{ijk})^2 = U_{cde} U_{fgh} U_{ijk} = 1,$$

10.62 $$U_{ghi}(ij) = (ij)\, U_{ghj},$$

10.63 $$V_{ghi}(ij) = (ij)\, V_{ghj} = V_{ghj} V_{ghi},$$

10.64 $$V_{cde} V_{fgh} = V_{fgh} V_{ijk}.$$

Note that $U_{cde} U_{fgh} U_{ijc}$ simply increases x_c and diminishes x_k, each by 3.

10.7. Now let i, j, k be any three different numbers among 1, 2, 3, 4, 5, 6, 7.

The transformation 10.13 gives the following correlation between the symmetries of

$$(\mathrm{PA})_9\, 2\sqrt{2} \qquad \text{and} \qquad (\mathrm{PA})_9\, 3\sqrt{2}:$$

$(\mathrm{PA})_9\, 2\sqrt{2}$	$(\mathrm{PA})_9\, 3\sqrt{2}$
$U_{ij}\, U_{k8}\, R^{-1}$	U_{ijk}
$U_{ij}\, R^{-1}$	U_{ij8}
$U_{ij}\, U_{j8}^{-1}$	U_{i89}
U_{ij}	U_{ij9}
U_{j8}	$U_{i89}^{-1}\, U_{ij9}$
R	$U_{ij8}^{-1}\, U_{ij9}$
(ij)	(ij)
$(i8)\, T_{i8}$	$(i8)$
S	(89)
$T_{i8}\, ST_{i8}$	$(i9)$
$T_{ij}\, ST_{ij}$	V_{ij8}
$T_{ijk8}\, ST_{ijk8} = [efgh\,.\,ijk8]$	V_{ijk}
$(i8)$	V_{i89}
$(ij)\, T_{ij}$	V_{ij9}
T_{ij}	$(ij)\, V_{ij9}$
T_{i8}	$(i8)\, V_{i89}$
ST	$V_{124}\, V_{235}\, V_{346}\, V_{457}\, V_{561}\, V_{672}\, V_{713}$

The factors in this last product of seven V's are all permutable, by **10.61**, since every pair of them have just one suffix-number in common. It is interesting that V_{ijk} appears as an extension of the bifid reflection.

In verification of 9.22, we have:

when $i, j < 8$,

$$\begin{aligned}\{(89)\,(ij)\,V_{ij9}\}^3 &= (ij)^3\,.\,(89)\,V_{ij9}\,(89)\,.\,V_{ij9}\,(89)\,V_{ij9}\\ &= (ij)\,.\,V_{ij8}\,.\,V_{ij8}\\ &= (ij),\end{aligned}$$

and when $j = 8$,

$$\begin{aligned}\{(89)\,(i8)\,V_{i89}\}^3 &= (89)\,.\,(i8)\,V_{i89}\,(i8)\,.\,(i9)\,V_{i89}\,(i9)\,.\,(89)\,V_{i89}\\ &= (89)\,.\,V_{i89}\,.\,V_{i89}\,.\,(89)\,V_{i89}\\ &= V_{i89}.\end{aligned}$$

10.8. Let a, b, c be three different numbers among 1, 2, 3, 4, 5, 6, 7, 8, 9 ; d, e, f, three different numbers among 1, 2, 3, 4, 5, 6, 7 ; g, h, i, three different numbers among 1, 2, 3, 4, 5, 6 ; j, j', two of 1, 2, 3 ; k, k', two of 4, 5, 6 ; l, l', two of 7, 8, 9. Also let T denote the reflection in the origin (*i.e.*, the simultaneous change of sign of *all* the co-ordinates.

With this notation, the chief symmetries of the polytopes under consideration may be tabulated as follows :—

Polytope.	Co-ords.	Symmetries.	No. of cases.
$(PA)_9\ 3\sqrt{2}$	10·1	(ab), V_{abc}, U_{abc}	36 + 84 + 84
$(PA)_8\ 3\sqrt{2}$	10·21	(ab), V_{abc}	36 + 84
$(PA)_7\ 3\sqrt{2}$	10·22	(de), V_{def}, V_{f89}	21 + 35 + 7
$(PA)_7\ 6\sqrt{2}$	10·23	(gh), (ll'), V_{ghl}	15 + 3 + 45
$(PA)_6\ 3\sqrt{2}$	10·24	(gh), V_{ghi}, V_{789}*	15 + 20 + 1
$(PA)_6\ 3\sqrt{2}$	10·25	(jj'), (kk'), (ll'), V_{jkl}	9 + 27
$(IA)_6\ 3\sqrt{2}$	10·32	(jj'), (kk'), (ll'), V_{jkl}, T	9 + 27 + 1
$(IA)_7\ 3\sqrt{2}$	10·31	(jj'), (kk'), (ll'), V_{jkl}, T, U_{jkl}	9 + 27 + 1 + 27

10.9. Below are summarized the most convenient co-ordinates for each of the polytopes 7.45.

$$2_{21}\ 3\sqrt{2} = (PA)_6\ 3\sqrt{2} : \begin{cases} (0, 0, 0 ;\ 2, -1, -1 ;\ 1, 1, -2), \\ (1, 1, -2 ;\ 0, 0, 0 ;\ 2, -1, -1), \\ (2, -1, -1 ;\ 1, 1, -2 ;\ 0, 0, 0). \end{cases}$$

$$1_{22}\ 3\sqrt{2} = (IA)_6\ 3\sqrt{2} : \begin{cases} (2, -1, -1 ;\ 2, -1, -1 ;\ 2, -1, -1), \\ (3, 0, -3 ;\ 0, 0, 0 ;\ 0, 0, 0), \\ (0, 0, 0 ;\ 3, 0, -3 ;\ 0, 0, 0), \\ (0, 0, 0 ;\ 0, 0, 0 ;\ 3, 0, -3), \\ (1, 1, -2 ;\ 1, 1, -2 ;\ 1, 1, -2). \end{cases}$$

$$3_{21}\ 4\sqrt{2} = (PA)_7\ 4\sqrt{2} : \begin{cases} (3, 3, -1, -1, -1, -1, -1, -1), \\ (1, 1, 1, 1, 1, 1, -3, -3). \end{cases}$$

$$2_{31}\ 2\sqrt{2} = (SA)_7\ 2\sqrt{2} : \begin{cases} (2, 0, 0, 0, 0, 0, 0, -2), \\ (1, 1, 1, 1, -1, -1, -1, -1). \end{cases}$$

$$1_{32}\ 4\sqrt{2} : \begin{cases} (7, -1, -1, -1, -1, -1, -1, -1), \\ (5, 1, 1, 1, 1, -3, -3, -3), \\ (3, 3, 3, -1, -1, -1, -1, -5), \\ (1, 1, 1, 1, 1, 1, 1, -7). \end{cases}$$

* By 10.64, $V_{789} = V_{123}\, V_{456}\, V_{123}$.

$4_{21}\, 3\sqrt{2} = (\mathrm{PA})_8\, 3\sqrt{2}$: $\begin{cases} (2, 2, 2, -1, -1, -1, -1, -1, -1), \\ (3, 0, 0, 0, 0, 0, 0, 0, -3), \\ (1, 1, 1, 1, 1, 1, -2, -2, -2). \end{cases}$

$2_{41}\, 3\sqrt{2}$: $\begin{cases} (5, 2, -1, -1, -1, -1, -1, -1, -1), \\ (4, 1, 1, 1, 1, -2, -2, -2, -2), \\ (3, 3, 0, 0, 0, 0, 0, -3, -3), \\ (2, 2, 2, 2, -1, -1, -1, -1, -4), \\ (1, 1, 1, 1, 1, 1, 1, -2, -5). \end{cases}$

$1_{42}\, 3\sqrt{2}$: $\begin{cases} (8, -1, -1, -1, -1, -1, -1, -1, -1), \\ (7, 1, 1, 1, -2, -2, -2, -2, -2), \\ (6, 3, 0, 0, 0, 0, -3, -3, -3), \\ (5, 5, -1, -1, -1, -1, -1, -1, -4), \\ (5, 2, 2, 2, -1, -1, -1, -4, -4), \\ (3, 3, 3, 3, 0, -3, -3, -3, -3), \\ (4, 4, 1, 1, 1, -2, -2, -2, -5), \\ (4, 1, 1, 1, 1, 1, 1, -5, -5), \\ (3, 3, 3, 0, 0, 0, 0, -3, -6), \\ (2, 2, 2, 2, 2, -1, -1, -1, -7), \\ (1, 1, 1, 1, 1, 1, 1, 1, -8). \end{cases}$

$5_{21}\, 3\sqrt{2} = (\mathrm{PA})_9\, 3\sqrt{2}$: 9 co-ordinates, mutually congruent modulo 3, sum zero.

$2_{51}\, 3\sqrt{2}$: The same, but with 4 or 8 of them *odd*.

$1_{52}\, 3\sqrt{2}$: 9 co-ordinates,* mutually congruent modulo 3, sum zero, satisfying the following further conditions :—

(*a*) If the co-ordinates are divisible by 3, then the three possible residues modulo 9 (namely 0, 3, 6) all occur (for the co-ordinates of each such point), one of them only once and the other two four times each.

(*b*) If not, then the residues modulo 9 either are all equal or take one of the following forms:

$(4^3, 1^6)$, $(1^3, 7^6)$, $(7^3, 4^6)$,
$(5^6, 2^3)$, $(2^6, 8^3)$, $(8^6, 5^3)$,
$(7^3, 4^3, 1^3)$, $(8^3, 5^3, 2^3)$.

* For these co-ordinates I am indebted to Mr. P. Du Val.

$3_{31}\, 2\sqrt{2} = (SA)_8\, 2\sqrt{2}$: 8 co-ordinates, mutually congruent modulo 2, sum zero.

$1_{33}\, 2\sqrt{2}$: The same, but with the further condition that the residues modulo 4 of the eight co-ordinates consist of two tetrads, such that the residues for each tetrad are equal among themselves, those for different tetrads not necessarily being different. (Thus the residues must consist of either eight 0's, 1's, 2's, 3's, four 0's and four 2's, or four 1's and four 3's.)

$2_{22}\, 3\sqrt{2} = (IA)_7\, 3\sqrt{2}$: 9 co-ordinates, mutually congruent modulo 3, falling into three definite triads (the same triads for every point) each with sum zero.

These co-ordinates can be verified by the method of 6.1, the work being simplified by the consideration that, if $p \neq q$, p_{qn} possesses all the symmetries of n_{pq}.

11. *Groups Generated by Two Operations.*

11.1. The group of symmetries of each existent polytope of the form

$$n_{pq} \qquad (n \geq -1),$$

except

$$(-1)_{pp} \qquad (= [\alpha_p, \alpha_p]),$$

can be generated by means of two or three special symmetries, two or three according as the polytope is finite or degenerate. We shall prove this by considering each case in detail, with the help of the following two general principles.

11.11. If certain given symmetries of the vertex figure $(n-1)_{pq}$ of a given polytope n_{pq} are known to generate the whole group of symmetries of $(n-1)_{pq}$, and are expressible in terms of certain symmetries X, X′, etc., of n_{pq} ; and if X, X′, etc., suffice to change any vertex of n_{pq} into any other ; then X, X′, etc., will generate the whole group of symmetries of n_{pq}.

11.12. If two symmetries, X and X′, of a given polytope which differs from a second given polytope only by lacking a certain symmetry Y of period 2 (*i.e.*, such that $Y^2 = 1$), are known to generate the whole group of symmetries of the first polytope ; and if X, of odd order (say h), is permutable with Y ; then (since $(XY)^h = Y$ and $(XY)^{h+1} = X$) the two symmetries XY and X′ will generate the whole group of symmetries of the second polytope.

11.2. The group of symmetries of

$$n_{p0} = \alpha_m \qquad (m = n + p + 1),$$

being simply the "symmetric group" on $m+1$ symbols (which can be taken to represent the vertices), is generated by any two cyclic permutations which "overlap" (*i.e.*, which have at least one symbol in common, the common symbols, if more than one, being arranged consecutively in the same order in the two cyclic permutations) and together involve all the symbols, without both involving an odd number of symbols. Thus the simplest generation is by

11.21 $$(12 \ldots r) \quad \text{and} \quad (m\; m+1),$$

where $r = m$ or $m+1$ (either, equally well). Another suitable pair of symmetries is

11.22 $$(12 \ldots r) \quad \text{and} \quad (m-1\; m\; m+1),$$
where
$$r = 2\left[\frac{m}{2}\right] \quad \text{or} \quad 2\left[\frac{m+1}{2}\right].*$$

The truncation
$$0_{np} = t_n \alpha_m$$
has precisely the same group of symmetries as α_m, except when $n=p$, *i.e.*, $m = 2n+1$.
$$0_{nn} = t_n \alpha_{2n+1}$$
possesses in addition the reflection in its centre, which we shall call T. By 11.12 and 11.21, its group of symmetries is generated by

11.23 $$(12 \ldots r)\mathrm{T} \quad \text{and} \quad (m\; m+1),$$
where
$$r = 2\left[\frac{m}{2}\right] + 1.$$

Alternatively, by 11.22, it is generated by

11.24 $$(12 \ldots r) \quad \text{and} \quad (m-1\; m\; m+1)\,\mathrm{T},$$
where
$$r = 2\left[\frac{m}{2}\right] \quad \text{or} \quad 2\left[\frac{m+1}{2}\right].$$

11.3. If
$$p \neq q,$$
the group of symmetries of
$$(-1)_{pq} = [\alpha_p, \alpha_q]$$
(which is the "direct product" of symmetric groups on $p+1$ symbols a_i and $q+1$ symbols b_j) is generated by

11.31 $$(a_1 a_2 \ldots a_r)(b_q\, b_{q+1}) \quad \text{and} \quad (b_1 b_2 \ldots b_s)(a_p\, a_{p+1}),$$
where
$$r = 2\left[\frac{p}{2}\right] + 1 \quad \text{and} \quad s = 2\left[\frac{q}{2}\right] + 1.$$

* "$\left[\frac{m}{2}\right]$" means "the greatest integer not greater than $\frac{m}{2}$."

For, calling these two operations A and B, we have

$$A^r = (b_q b_{q+1}) \qquad \text{and} \qquad B^s = (a_p a_{p+1}),$$

$$A^{r+1} = (a_1 a_2 \dots a_r) \quad \text{and} \quad B^{s+1} = (b_1 b_2 \dots b_s).$$

Thus A^{r+1} and B^s give all the permutations of the a's, while B^{s+1} and A^r give all the permutations of the b's.

If p and q were equal, we should require a further operation (of period 2) to interchange the a's and b's bodily.

11.4. The group of symmetries of

$$n_{11} = \beta_m \qquad (m = n+3)$$

is generated by the permutations of the vertices of one bounding α_{m-1}, together with the reflection (T_1, say) which simply interchanges one pair of opposite vertices. By 11.12 and 11.22, the group is therefore generated by

11.41 $$(12 \dots r) \quad \text{and} \quad (m-2\ m-1\ m)\, T_1,$$

where

$$r = 2\left[\frac{m-1}{2}\right] \quad \text{or} \quad 2\left[\frac{m}{2}\right].$$

In considering

$$1_{n1} = h\gamma_m,$$

we shall use the co-ordinates 6.22. Let $(12 \dots r)$ denote the cyclic permutation of the co-ordinates $x_1, x_2, \dots x_r$; and let T_{ij} subtract the co-ordinates x_i, x_j each from 1. Then clearly the group of symmetries is generated by

11.42 $$(12 \dots r) \quad \text{and} \quad (m-2\ m-1\ m)\, T_{12},$$

where r is the same as in 11.41.

When $m = 5$, there is an interesting alternative generation. Let

$$[ij] = (ij)\, T_{ij}.$$

Then, since

$$(12) = [14]\,[24]\,[14]$$

and

$$[14] = (12345)^2\,[24]\,(12345)^{-2},$$

the group of symmetries of

$$(\text{PA})_5 = 1_{21} = h\gamma_5$$

is generated by

11.43 $$(12345) \quad \text{and} \quad [24].$$

11.5. The principle 11.11 now proves that the group of symmetries of

$$(\text{PA})_6 = 2_{21}$$

is generated by the *three* operations

11.51 $$(12345), \quad (56) \quad \text{and} \quad [135\,.\,246]$$

in the notation of 9.44. For, these symmetries suffice to change any vertex into any other, while the (magnified) vertex figure 9.45 possesses the symmetries 11.43, which are related to 11.51 by the identity

$$[24] = [135\,.\,246].$$

In the next section (11.6) we shall reduce these three symmetries to two.

The same principle proves that the group of symmetries of

$$(\mathrm{PA})_7 = 3_{21}$$

is generated by the two,

11.52 $$(1234567) \quad \text{and} \quad [1357\,.\,2468],$$

in the notation of 9.34. For, the symmetries 11.52 together give [2461 . 3578], this and [1357 . 2468] give (by 9.35) the transposition (18), this and (1234567) give all the permutations of 1, 2, 3, 4, 5, 6, 7, 8, and these with [1357 . 2468] give all the bifid reflections. Finally, we can express the symmetries 11.51 of the vertex figure in terms of those we have just found, in virtue of 9.37.

Incidentally, it follows that the group (of order $\frac{1}{2}\,7\,!\;P_7 = 1451520$) of automorphisms of the 28 bitangents c_{ij} of the general plane quartic curve, can be generated by means of the cyclic permutation

$$(1234567)$$

of seven of the eight suffix-numbers, together with the "bifid substitution"

$$1357\,.\,2468.$$

(Regarded as a symmetry of $(\mathrm{PA})_7$, the latter is the "bifid rotation" which, in our notation, takes the value

$$[1357\,.\,2468]\,\mathrm{T} = \mathrm{T}_{1357}\,\mathrm{ST}_{2468}.)$$

11.6. In 10.8, we saw that

$$(\mathrm{PA})_6\, 3\sqrt{2},$$

in the form 10.25, possesses the 9 + 27 symmetries

11.61 $$(jj'), \quad (kk'), \quad (ll'), \quad \mathrm{V}_{jkl}\,;$$

where

j, j' are two of 1, 2, 3 ; k, k', two of 4, 5, 6 ; l, l', two of 7, 8, 9.

Now, the points 10.25 or 10.27 can be actually transformed into the points 9.41 or 9.43, by means of the transformation

$$2\Omega\mathrm{T}_{123}\,\mathrm{RU}_{78}\,\tfrac{1}{3},$$

where Ω is defined in 10.12, R and U_{78} in 9.1, while T_{123} changes the signs of x_1, x_2 and x_3. This means that we take in turn each point of 10.25, double all its co-ordinates, operate with Ω, change the first three signs, drop the ninth co-ordinate (which is now constantly zero), add one to each of the remaining co-ordinates and a further two to to the seventh and eighth, and finally divide throughout by 3. Explicitly, in virtue of 10.11, the relation

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9)\, 2\Omega T_{123}\, RU_{78}\, \tfrac{1}{3} = (x'_1, x'_2, x'_3, x'_4, x'_5, x'_6, x'_7, x'_8)$$

implies

$$\begin{cases} x'_j = \tfrac{1}{3}(1 - 2x_j - x_9) & (j = 1 \text{ or } 2 \text{ or } 3), \\ x'_k = \tfrac{1}{3}(1 + 2x_k + x_9) & (k = 4 \text{ or } 5 \text{ or } 6), \\ x'_l = \tfrac{1}{3}(3 + 2x_l + x_9) & (l = 7 \text{ or } 8). \end{cases}$$

The symmetries 11.61 become (by the same transformation) the 15 + 1 + 20 symmetries,

11.62 (ij), (78), $[fgh\,.\,ijk]$ $(f, g, h, i, j, k = 1, 2, 3, 4, 5, 6)$,

of the

$$(PA)_6\, 2\sqrt{2}$$

whose vertices are 9.43 ; (ij) being a transposition of the suffix-numbers, (78) the transposition of a and b, and $[fgh\,.\,ijk]$ the operation 9.44.

The details of the correlation are as follows :

Vertices.

$u_1 s_j$	a_j
$u_2 s_j$	b_j
$u_3 s_j$	$c_{j'j''}$
$s_j t_{k-3}$	c_{jk}
$t_{k-3} u_1$	b_k
$t_{k-3} u_2$	a_k
$t_{k-3} u_3$	$c_{k'k''}$

Symmetries.

(jj')	(jj')
(kk')	(kk')
(78)	(78)
(79)	$[123\,.\,456]$
(89)	$[456\,.\,123]$
V_{jk7}	$[j'j''k\,.\,jk'k'']$
V_{jk8}	$[jk'k''\,.\,j'j''k]$
V_{jk9}	(jk)

Here

$$j, j', j'' \quad \text{are} \quad 1, 2, 3 \quad \text{in any order,}$$

and

$$k, k', k'' \quad \text{are} \quad 4, 5, 6 \quad \text{in any order.}$$

We have seen that the whole group of symmetries of

$$(\mathrm{PA})_6 = 2_{21}$$

is generated by 11.51 and *a fortiori* by 11.62. Hence it is also generated by 11.61, and consequently by the *two* symmetries

11.63 $$(147258369) \quad \text{and} \quad \mathrm{V}_{169}.$$

For, we at once obtain

$$\mathrm{V}_{149}, \mathrm{V}_{147}, \mathrm{V}_{247}, \mathrm{V}_{257}, \mathrm{V}_{258}, \mathrm{V}_{358}, \mathrm{V}_{368}, \mathrm{V}_{369};$$

and thence, by 10.63, all the transpositions

$$(64), (97), (12), (45), (78), (23), (56), (89), (31);$$

which in turn give the rest of the V_{jkl}'s.

Incidentally, it follows that the group (of order $6!\ \mathrm{P}_6 = 51840$) of automorphism of the 27 lines on the general cubic surface, can be generated by two operations. For details, see the Appendix.

11.7. Since

$$(1234567) = (12)(23)(34)(45)(56)(67)$$

and

$$(ij) = (\mathrm{ST}_{ij})^3$$

and

$$[1357 \,.\, 2468] = \mathrm{T}_{13}\,\mathrm{T}_{57}\,\mathrm{ST}_{13}\,\mathrm{T}_{57},$$

it follows (by 11.11 and 11.52) that the group of symmetries of

$$(\mathrm{PA})_8 = 4_{21}$$

is generated by

$$\mathrm{S} \quad \text{and} \quad \mathrm{T}_{ij} \qquad (i, j = 1, 2, 3, 4, 5, 6, 7, 8;\ i \neq j),$$

in the notation of 9.2.

The correlation 10.7 changes these particular symmetries into

$$(89) \quad \text{and} \quad (ij)\,\mathrm{V}_{ij9}.$$

But, by 10.63,

$$(ij) = \mathrm{V}_{ghi}\,\mathrm{V}_{ghj}\,\mathrm{V}_{ghi}.$$

Hence the group is generated by

$$\mathrm{V}_{ijk} \qquad (i, j, k = 1, 2, 3, 4, 5, 6, 7, 8;\ i \neq j \neq k \neq i),$$

where the suffix-number 9 may be excluded, in virtue of 10.64. Finally, it is generated by the two symmetries

11.71 $$(12345678) \quad \text{and} \quad V_{123}.$$

For, these together give

$$V_{234}, \quad V_{456}, \quad V_{567}, \quad V_{781}, \quad V_{812};$$

whence we obtain the particular transpositions

$$(14), \quad (47), \quad (72).$$

Since

$$(12) = (72)\,(47)\,(14)\,(47)\,(72),$$

we can deduce all the permutations of 1, 2, 3, 4, 5, 6, 7, 8, and thence the rest of the V_{ijk}'s.

It follows now from 11.11 that the group of symmetries of

$$(PA)_9 = 5_{21}$$

is generated by

11.72 $$(12345678) \quad \text{and} \quad V_{123} \quad \text{and} \quad U_{123},$$

in the notation of 10.6.

11.8. From the point of view of symmetry, $(PA)_6$ differs from its semi-reciprocal

$$(IA)_6 = 1_{22}$$

only by lacking the reflection in its centre. Hence, by 11.12 and 11.63, the group of symmetries of the latter polytope is generated by

11.81 $$(147258369)\,T \quad \text{and} \quad V_{169}.$$

For the degenerate

$$(IA)_7 = 2_{22},$$

we have to insert a translation. So its group of symmetries is generated by

11.82 $$(147258369)\,T \quad \text{and} \quad V_{169} \quad \text{and} \quad U_{169}.$$

(It seems possible that the " T " is here unnecessary, but this has not been proved.)

2_{31} and 1_{32} have the same symmetries (11.52) as $(PA)_7 = 3_{21}$.

2_{41} ,, 1_{42} ,, ,, ,, (11.71) ,, $(PA)_8 = 4_{21}$.

2_{51} ,, 1_{52} ,, ,, ,, (11.72) ,, $(PA)_9 = 5_{21}$.

By 11.11, the group of symmetries of

$$(SA)_8 = 3_{31}$$

is generated by 11.52 along with a suitable translation, in fact by

11.83 $$(1234567) \quad \text{and} \quad T_{1357}\,ST_{1357} \quad \text{and} \quad T_{1357}\,RT_{1357}$$

(see 9.6).

Now only 1_{33} remains to be examined.

11.9. By 7.8, the vertices of $1_{33}\times$ are the centres of the α_7's of 3_{31}.

Consider the $(SA)_8\ 8\surd 2$ whose vertices have eight co-ordinates, all congruent to 0 or to 4, modulo 8, and adding up to zero. The centre of a typical bounding $\alpha_7\ 8\surd 2$ is

11.91 $$(1, 1, 1, 1, 1, 1, 1;\ -7).$$

In virtue of the symmetries

$$(1234567) \quad \text{and} \quad T_{1357}\,ST_{1357} \quad \text{and} \quad T_{1357}\,R^4\,T_{1357},$$

this point gives rise to the totality of points whose eight co-ordinates, adding up to zero, have, as residues modulo 8, either eight 1's, 3's, 5's, 7's, four 1's and four 5's, or four 3's and four 7's. Since these points possess the additional symmetry T′* which reflects in the point 11.91, they must be the vertices of $1_{33}\times$ (in fact, of $1_{33}\ 4\surd 2$).

Hence the group of symmetries of

$$1_{33}$$

is generated by

11.92 $$(1234567)\,T' \quad \text{and} \quad T_{1357}\,ST_{1357} \quad \text{and} \quad T_{1357}\,R^4\,T_{1357}.$$

12. *Truncations of n_{pq}. Tables.*

12.1. The only non-trivial polytopes of the form 7.31 are

12.11 $$O_{npq} = [\alpha_n,\ \alpha_p,\ \alpha_q]^{+1},$$

with the existence condition 7.32. For otherwise, the simplest possible vertex figure which does not reduce to a prism with only two constituents is

$$[\alpha_1,\ \alpha_2,\ \beta_q],$$

which has circum-radius

$$\surd(\tfrac{1}{4} + \tfrac{1}{3} + \tfrac{1}{2}) > 1.$$

Clearly

12.12 $$O_{0pq} = 0_{pq} = t_p\,\alpha_{p+q+1}$$

and

12.13 $$O_{n11} = [\alpha_n,\ \beta_2]^{+1} = t_2\,\gamma_{n+3}.$$

In particular,

$$O_{000} = \alpha_1$$

and

$$O_{111} = \{3, 4, 3\}.$$

Thus the only new polytopes which arise in this way are

12.14

O_{221}
O_{321}
O_{421}
O_{521}
O_{331}
O_{222}

of which the last three are degenerate (by 7.46).

* In terms of the usual symbols,

$$T' = (U_{18}U_{28}U_{12}^{-1})^2\,R^{-1}TR\,(U_{12}U_{28}^{-1}U_{18}^{-1})^2.$$

As in the case of n_{pq}, the number of dimensions is

$$m = n + p + q + 1.$$

12.2. By 7.6, the number of vertices of O_{npq} is

12.21 $$({}^{0}|_{m}) = [n\,p\,q]$$

Also, by 4.6, the elements of

$$[\alpha_n, \alpha_p, \alpha_q]$$

consist of

$$\binom{n+1}{n'+1}\binom{p+1}{p'+1}\binom{q+1}{q'+1} \qquad [\alpha_{n'}, \alpha_{p'}, \alpha_{q'}]\text{'s},$$

for all n', p', q' satisfying

$$\begin{cases} 0 \leq n' \leq n, \\ 0 \leq p' \leq p, \\ 0 \leq q' \leq q. \end{cases}$$

Hence, by 2.52, the elements, other than vertices, of O_{npq}, consist of

12.22 $$({}^{n'+p'+q'+1}|_{m}) = \binom{n+1}{n'+1}\binom{p+1}{p'+1}\binom{q+1}{q'+1}\frac{[n\,p\,q]}{[n'p'q']}$$

$$\mathrm{O}_{n'p'q'}\text{'s}.$$

As in the case of n_{pq}, we fix the order of the suffixes so as to distinguish between equal elements (such as O_{rst} and O_{srt}, $r \neq s$) which are of different type. In the case of O_{222}, equal elements are always equivalent, but the division into types indicates that various kinds of elements can be divided uniquely into three (indeed the $t_1\alpha_4$'s into six) congruent sets.

By 2.41 and 4.71, the order of the group of symmetries of O_{npq} is

12.23 $$g_m = \lambda(n+1)!\,(p+1)!\,(q+1)!\,[n\,p\,q],$$

where

$$\lambda = 1 + \varepsilon_{pq} + \varepsilon_{qn} + \varepsilon_{np} + 2\varepsilon_{pq}\,\varepsilon_{qn}\,\varepsilon_{np} - \varepsilon_{p0}\,\varepsilon_{q0} - \varepsilon_{q0}\,\varepsilon_{n0} - \varepsilon_{n0}\,\varepsilon_{p0} - 2\varepsilon_{n0}\,\varepsilon_{p0}\,\varepsilon_{q0}.$$

12.3. We saw in 7.35 that

12.31 $$\mathrm{O}_{npq} = t_n n_{pq}.$$

There are, by 5.8, other truncations

$$t_l n_{pq}$$

for all

$$l < n$$

and also (since all $(n+1)$-dimensional elements of n_{pq} are of type n_{00}) for

$$l = n + 1.$$

12.4. One naturally tries to obtain some sort of higher truncations by taking for vertices the centres of all those elements of n_{pq} which are of the same type

$$n_{p'q'} \qquad (p' + q' > 0).$$

The results are as follows :—

If $q' = q$ we obtain $t_{p-p'-1}\, p_{qn}$ (by the theorem stated at the end of 7.8).

Similarly, if $p' = p$ we obtain $t_{q-q'-1}\, q_{np}$.

But if $p > p' \neq q' < q$, the resulting polytope is not uniform, its edges being unequal. (The case $p' = q'$ offers a new field for research.)

To take a very simple example, consider the edges of the triangular prism $(-1)_{21}$. The centres of the lateral edges, which are of type $(-1)_{01}$ (since they do not belong to the base $(-1)_{20}$), are the vertices of $t_1 2_{1\,(-1)} = \alpha_2$. But the centres of the basal edges, of type $(-1)_{10}$, are the vertices of the *thin* triangular prism $[\alpha_2 \tfrac{1}{2},\ \alpha_1]$.

12.5. By 7.72, if $l \leq n$, $t_l n_{pq}$ has $\binom{n+1}{l+1} \dfrac{[n\ p\ q]}{[(n-l-1)\ p\ q]}$ vertices. By 5.8, its vertex figure is $[\alpha_l,\ (n-l-1)_{pq}]$.

But, by 7.73, $t_{n+1} n_{pq}$ has $(p+1)(q+1)\ \dfrac{[n\,p\,q]}{n+2}$ vertices. Its vertex figure is $[\alpha_{n+1},\ (\alpha_{p-1} \overline{\ \ \ \sqrt{2}\ \ \ } \alpha_{q-1})]$.

These facts are sufficient to determine all the numerical properties of

$$t_l n_{pq} \qquad (l \leq n+1).$$

In particular, it is bounded by

$$(n+1)\,\frac{[n\ p\ q]}{[(n-1)\,p\,q]} \qquad t_{l-1}\,(n-1)_{pq}\text{'s}$$

and

$$(p+1)\,\frac{[n\ p\ q]}{[n\ (p-1)\ q]} \qquad t_l\, n_{(p-1)\,q}\text{'s}$$

and

$$(q+1)\,\frac{[n\ p\ q]}{[n\ p\ (q-1)]} \qquad t_l\, n_{p\,(q-1)}\text{'s}.$$

12.6. In the following tables, the elements of each polytope are given in a column. The numbers referring to equal elements of different type are bracketed. In 12.7 and 12.8, the type-symbols are given immediately after the numbers ; in 12.9 they appear at the ends of the lines. In 12.7 and 12.8, the numbers unaccompanied by type-symbols refer to α's in the first category of 7.5 ; in 12.9, such numbers refer to vertices (*i.e.*, every line ends in a type-symbol except that headed " α_0 ").

Everything in these tables is deducible from 7.4, 7.7, 7.9, 12.1 and 12.2, the values of

$$[n\ p\ q]$$

being (by 7.6 and 8.7)—

$$[(-1)\ p\ q] = 1,$$

$$[0\ p\ q] = \binom{p+q+2}{p+1},$$

$$[n\ 1\ 1] = 2^n (n+2)(n+3),$$

$$[2\ 2\ 1] = 720,$$

$$[3\ 2\ 1] = 10080,$$

$$[4\ 2\ 1] = 483840,$$

$$[5\ 2\ 1] = [3\ 3\ 1] = [2\ 2\ 2] = \infty\,.$$

12.7. Table of the Simpler Polytopes n_{pq}, namely, those belonging to Infinite Series of Polytopes.

Name :		$(-1)_{pq} = [\alpha_p, \alpha_q]$		$0_{pq} = t_p\alpha_{p+q+1}$		$1_{n1} = h\gamma_{n+3}$		$n_{11} = \beta_{n+3}$	
Order of group :		$(1+\varepsilon_{pq})(p+1)!\,(q+1)!$		$(1+\varepsilon_{pq})(p+q+2)!$		$(1+\varepsilon_{n1})\,2^{n+2}(n+3)!$		$2^{n+3}(n+3)!$	
No. of dimensions :		$p+q$		$p+q+1$		$n+3$		$n+3$	
(Vertices)	α_0	$(p+1)(q+1)$	$(-1)_{00}$	$\binom{p+q+2}{p+1}$		2^{n+2}		$2(n+3)$	
(Edges)	α_1	$\binom{p+1}{2}(q+1)$	$(-1)_{10}$	$\binom{p+q+2}{p}\binom{q+2}{2}$	0_{00}	$2^n(n+2)(n+3)$		$2(n+2)(n+3)$	
		$(p+1)\binom{q+1}{2}$	$(-1)_{01}$						
	α_{r+2}					$2^{n+2}\binom{n+3}{r+3}$	1_{r0}	$2^{r+3}\binom{n+3}{r+3}$ $(r<n-1)$	
	α_{n+1}							$2^{n+2}(n+3)$	n_{00}
	α_{n+2}							2^{n+2}	n_{10}
								2^{n+2}	n_{01}
	$[\alpha_r, \alpha_s]$	$\binom{p+1}{r+1}\binom{q+1}{s+1}$	$(-1)_{rs}$						
		$\binom{p+1}{s+1}\binom{q+1}{r+1}$	$(-1)_{sr}$						
	$t_r\alpha_{r+s+1}$			$\binom{p+q+2}{p-r}\binom{r+q+2}{q-s}$	0_{rs}				
				$\binom{p+q+2}{p-s}\binom{s+q+2}{q-r}$	0_{sr}				
	$h\gamma_{r+3}$					$2^{n-r}\binom{n+3}{r+3}$	1_{r1}		

N.B.—This table includes the properties of :—

$(PA)_3 = (-1)_{21}$, $(PA)_4 = 0_{21}$, $(PA)_5 = 1_{21}$,
$(SA)_4 = (-1)_{31}$, $(SA)_5 = 0_{31}$, $(SA)_6 = 1_{31}$,
$(IA)_4 = (-1)_{22}$, $(IA)_5 = 0_{22}$.

12.8. Table of Special

Name :	$2_{21}=(PA)_6$		$1_{22}=(IA)_6$		$3_{21}=(PA)_7$		$2_{31}=(SA)_7$		1_{32}		$4_{21}=(PA)_8$	
Order of group :	51840		103680		2903040		2903040		2903040		696729600	
No. of dimensions :	6		6		7		7		7		8	
(Vertices) α_0	27		72		56		126		576		240	
(Edges) α_1	216		720		756		2016		10080		6720	
α_2	720		2160	1_{00}	4032		10080		40320	1_{00}	60480	
			1080	1_{10}					30240	1_{10}		
α_3	1080	2_{00}			10080		20160	2_{00}			241920	
			1080	1_{01}					20160	1_{01}		
	432	2_{10}	216	1_{20}			12096	2_{10}	12096	1_{20}		
α_4					12096	3_{00}					483840	
	216	2_{01}	216	1_{02}			4032	2_{01}	4032	1_{02}		
β_4			270	1_{11}					7560	1_{11}		
					4032	3_{10}						
α_5	72	2_{20}					4032	2_{20}	2016	1_{30}	483840	4_{00}
					2016	3_{01}						
β_5	27	2_{11}					756	2_{11}				
$h\gamma_5$			27	1_{21}					1512	1_{21}		
			27	1_{12}					756	1_{12}		
											138240	4_{10}
α_6					576	3_{20}	576	2_{30}				
											69120	4_{01}
β_6					126	3_{11}						
$h\gamma_6$									126	1_{31}		
$(PA)_6 = 2_{21}$							56	2_{21}				
$(IA)_6 = 1_{22}$									56	1_{22}		
α_7											17280	4_{20}
β_7											2160	4_{11}
$h\gamma_7$												
$(PA)_7 = 3_{21}$												
$(SA)_7 = 2_{31}$												
1_{32}												
α_8												
β_8												
$h\gamma_8$												
2_{41}												
1_{42}												

Polytopes n_{pq}.

2_{41}		1_{42}		$5_{21}=(PA)_9$		2_{51}		1_{52}		$3_{31}=(SA)_8$		1_{33}		$2_{22}=(IA)_7$	
696729600		696729600		∞		∞		∞		∞		∞		∞	
8		8		9		9		9		8		8		7	
2160		17280		∞		∞		∞		∞		∞		∞	
69120		483840		∞		∞		∞		∞		∞		∞	
483840		2419200	1_{00}	∞		∞		∞	1_{00}	∞		∞	1_{00}	∞	
		2419200	1_{10}					∞	1_{10}			∞	1_{10}		
1209600	2_{00}			∞		∞	2_{00}			∞				∞	2_{00}
		1209600	1_{01}					∞	1_{01}			∞	1_{01}		
967680	2_{10}	1451520	1_{20}			∞	2_{10}	∞	1_{20}			∞	1_{20}	∞	2_{10}
				∞						∞	3_{00}				
241920	2_{01}	241920	1_{02}			∞	2_{01}	∞	1_{02}			∞	1_{02}	∞	2_{01}
		604800	1_{11}					∞	1_{11}			∞	1_{11}		
										∞	3_{10}	∞	1_{30}	∞	2_{20}
483840	2_{20}	483840	1_{30}	∞		∞	2_{20}	∞	1_{30}						
										∞	3_{01}	∞	1_{03}	∞	2_{02}
60480	2_{11}					∞	2_{11}							∞	2_{11}
		181440	1_{21}					∞	1_{21}			∞	1_{21}		
		60480	1_{12}					∞	1_{12}			∞	1_{12}		
138240	2_{30}	69120	1_{40}	∞	5_{00}	∞	2_{30}	∞	1_{40}	∞	3_{20}				
										∞	3_{11}				
												∞	1_{31}		
		30240	1_{31}					∞	1_{31}						
												∞	1_{13}		
														∞	2_{21}
6720	2_{21}					∞	2_{21}								
														∞	2_{12}
		6720	1_{22}					∞	1_{22}			∞	1_{22}		
				∞	5_{10}										
17280	2_{40}					∞	2_{40}	∞	1_{50}	∞	3_{30}				
				∞	5_{01}										
		2160	1_{41}					∞	1_{41}						
										∞	3_{21}				
240	2_{31}					∞	2_{31}								
												∞	1_{32}		
		240	1_{32}					∞	1_{32}						
												∞	1_{23}		
				∞	5_{20}	∞	2_{50}								
				∞	5_{11}										
								∞	1_{51}						
						∞	2_{41}								
								∞	1_{42}						

12.9. TABLE of Polytopes O_{npq}.

		Name :	O_{221}	O_{321}	O_{421}	O_{521}	O_{331}	O_{222}	Type
		Order of group :	103680	2903040	696729600	∞	∞	∞	
		No. of dimensions :	6	7	8	9	8	7	
(Vertices)	α_0		720	10080	483840	∞	∞	∞	
(Edges)	α_1		6480	120960	7257600	∞	∞	∞	O_{000}
	α_2		4320	120960	9676800	∞	∞	∞	O_{100}
			4320	80640	4838400	∞	∞	∞	O_{010}
			2160	40320	2419200	∞	∞	∞	O_{001}
	α_3		1080	60480	7257600	∞	∞	∞	O_{200}
			1080	20160	1209600	∞	∞	∞	O_{020}
								∞	O_{002}
	β_3		2160	60480	4838400	∞	∞	∞	O_{110}
			1080	30240	2419200	∞	∞	∞	O_{101}
			1080	20160	1209600	∞	∞	∞	O_{011}
	α_4			12096	2903040	∞	∞		O_{300}
							∞		O_{030}
	$t_1\alpha_4$		432	24192	2903040	∞	∞	∞	O_{210}
			432	12096	967680	∞	∞	∞	O_{120}
			216	12096	1451520	∞	∞	∞	O_{201}
			216	4032	241920	∞	∞	∞	O_{021}
								∞	O_{102}
								∞	O_{012}
	{3, 4, 3}		270	7560	604800	∞	∞	∞	O_{111}

α_5			483840	∞			O_{400}
$t_1\alpha_5$		4032	967680	∞	∞		O_{310}
					∞		O_{130}
		2016	483840	∞	∞		O_{301}
					∞		O_{031}
$t_2\alpha_5$	72	4032	483840	∞	∞	∞	O_{220}
						∞	O_{202}
						∞	O_{022}
$t_2\gamma_5$	27	1512	181440	∞	∞	∞	O_{211}
	27	756	60480	∞	∞	∞	O_{121}
						∞	O_{112}
α_6				∞			O_{500}
$t_1\alpha_6$			138240	∞			O_{410}
			69120	∞			O_{401}
$t_2\alpha_6$		576	138240	∞	∞		O_{320}
					∞		O_{230}
$t_2\gamma_6$		126	30240	∞	∞		O_{311}
					∞		O_{131}
O_{221}		56	6720	∞	∞	∞	O_{221}
						∞	O_{212}
						∞	O_{122}
$t_1\alpha_7$				∞			O_{510}
				∞			O_{501}
$t_2\alpha_7$			17280	∞			O_{420}
$t_3\alpha_7$					∞		O_{330}
$t_2\gamma_7$			2160	∞			O_{411}
O_{321}			240	∞	∞		O_{321}
					∞		O_{231}
$t_2\alpha_8$				∞			O_{520}
$t_2\gamma_8$				∞			O_{511}
O_{421}				∞			O_{421}

12.91 ANALOGOUS TABLE for $O_{n11} = t_2\gamma_{n+3}$.

Name :	O_{n11}	
Order of group :	$(1 + 2\varepsilon_{n1})\, 2^{n+3}\,(n+3)\,!$	
No. of dimensions :	$n + 3$	Type
(Vertices) α_0	$2^n\,(n+2)\,(n+3)$	
$\alpha_{n'+1}$	$2^{n+2}\,(n+3)\binom{n+2}{n'+2}$	$O_{n'00}$
$t_1\alpha_{n'+2}$	$2^{n+2}\binom{n+3}{n'+3}$	$O_{n'10}$
	$2^{n+2}\binom{n+3}{n'+3}$	$O_{n'01}$
$t_2\gamma_{n'+3}$	$2^{n-n'}\binom{n+3}{n'+3}$	$O_{n'11}$

APPENDIX

On the Generation of the Group of the Lines of a Cubic Surface by Two Operations.

13.1. The group of symmetries of $(PA)_6$ is simply isomorphic with the group of automorphisms of the lines on a general cubic surface. For, there is a perfect correspondence between the distances occurring among the vertices of $(PA)_6$ and the intersections occurring among the lines. This may easily be seen by comparing 9.43 with the ordinary SCHLÄFLI notation for the lines.

13.2. It is known* that the group is generated by the combination of every permutation of the suffix-numbers

$$1,\ 2,\ 3,\ 4,\ 5,\ 6$$

with any particular " bifid substitution " such as

$$[135\ .\ 246]$$

(defined in 9.44), and therefore, for example, by the *three* operations consisting of this bifid substitution and the two cyclic permutations

$$(16), \quad (123456).$$

* BURNSIDE, 'Proc. Lond. Math. Soc.,' vol. 10, p. 301 (1911).

13.3. In 11.63, we found (in terms of a different notation) a generation of the group by *two* operations. It is of interest to translate this into the familiar Schläfli notation.

Let

$$\begin{cases} H_0 = V_{169}, \\ \omega = (147258369)^4. \end{cases}$$

Then, since

$$\omega^7 = (147258369),$$

it must be possible to generate the group by means of H_0 and ω.

13.4. On reverting to the Schläfli notation, we find (by the correlation given in 11.6)—

$$H_0 = (16)$$

and

$$\omega = XYZ,$$

where X, Y, Z are cyclic permutations, each of nine of the twenty-seven lines, namely,

$$X = (a_3 b_6 c_{26}\ c_{13} c_{46} c_{15}\ b_1 a_4 c_{34}),$$
$$Y = (c_{23} a_5 c_{14}\ b_3 b_4 c_{36}\ a_2 c_{45} c_{25}),$$
$$Z = (b_2 c_{56} c_{35}\ a_1 a_6 c_{24}\ c_{12} b_5 c_{16}).$$

13.5. To verify that the group is generated by the single operation ω, of period 9, and the transposition H_0; it is sufficient to express, in terms of ω and H_0, five consecutive transpositions (thus providing all permutations of the six suffix-numbers) and the bifid substitution. Let

$$H_n = \omega^n H_0 \omega^{9-n}$$

(the operations written to the left being those *first* performed). We then have

$$(23) = H_6 H_8 H_6, \quad (36) = H_2, \quad (61) = H_0, \quad (14) = H_7, \quad (45) = H_3 H_1 H_3$$

and

$$[135 \,.\, 246] = H_1.$$

13.6. That ω is actually an operation of the group may be directly verified. Let the lines

$$a_3, \quad c_{23}, \quad b_2,$$

or any other three lines which occur in corresponding places in the three brackets X, Y, Z, written down above, be respectively denoted by

$$\xi, \eta, \zeta.$$

Then the three sets, each of nine lines, take the form

$$\xi\omega^i, \eta\omega^j, \zeta\omega^k \quad (i, j, k = 0, 1, 2, 3, 4, 5, 6, 7, 8).$$

If " $\sim$ " means "intersects," (remembering that $\omega^9 = 1$) the rules of intersection are as follows :—

$$\xi\omega^i \sim \eta\omega^i \sim \zeta\omega^i,$$
$$\xi\omega^i \sim \xi\omega^{i+3} \sim \xi\omega^{i+6},$$
$$\eta\omega^j \sim \eta\omega^{j+3} \sim \eta\omega^{j+6},$$
$$\zeta\omega^k \sim \zeta\omega^{k+3} \sim \zeta\omega^{k+6},$$

$$\xi\omega^i \sim \xi\omega^{i+1},$$
$$\eta\omega^j \sim \eta\omega^{j+2},$$
$$\zeta\omega^k \sim \zeta\omega^{k+4},$$
$$\eta\omega^i \sim \zeta\omega^{i+\lambda} \quad \text{if} \quad \lambda \equiv \pm 1 \pmod{9},$$
$$\zeta\omega^i \sim \xi\omega^{i+\mu} \quad \text{if} \quad \mu \equiv \pm 2 \pmod{9},$$
$$\xi\omega^i \sim \eta\omega^{i+\nu} \quad \text{if} \quad \nu \equiv \pm 4 \pmod{9}.$$

(The numbers ± 1, ± 2, ± 4 are the residues, modulo 9, of the various powers of 2, and the notation can be further elaborated.) These rules, it is easy to see, give the 135 intersections, which are therefore unaffected by ω. So ω belongs to the group, as required.

13.7. The operation ω may be regarded as a product PQ, in which each of P, Q is of period 2 ; Q being in fact

$$(16)(25)(34).$$

This is seen at once by applying to each of X, Y, Z (of which no two have a line in common) the obvious decomposition of a cyclic permutation of period 9,

$$(123456789) = (17)(26)(35)(89) \,.\, (18)(27)(45)(36).$$

13.8. It is manifest that instead of ω we may take any power of ω, say ω^n, where n is not a multiple of 3. Or we may take, instead of ω, an operation obtained from it by any permutation of the suffix-numbers 1, 2, 3, 4, 5, 6.

13.9. By 9.35,

$$[fgi \,.\, jhk][fgj \,.\, ihk][fgi \,.\, jhk] = (ij).$$

Beside the equation

$$H_1 = [135 \,.\, 246],$$

we have also

$$H_3 = [134 \,.\, 256],$$
$$H_4 = [345 \,.\, 126],$$
$$H_5 = [234 \,.\, 156],$$
$$H_6 = [346 \,.\, 125],$$
$$H_8 = [246 \,.\, 135].$$

Thus many identities, besides those utilised in 13.5, are obtainable ; as for instance

$$H_1H_4H_1 = H_7,$$
$$H_3H_5H_3 = (12),$$
$$(H_3H_5)^3 = 1.$$

Also, from the equations in 13.5, it is clear that, instead of the particular transposition (16), we might quite similarly have used (14) or (36).

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

TABLE of Symbols,

together with equivalent symbols used by previous authors (see Preface).

"A."—A. B. STOTT and P. H. SCHOUTE.
"B."—E. L. ELTE.
"C."—D. M. Y. SOMMERVILLE.

(The method adopted is to go through the Latin alphabet, then the Greek alphabet, and finally to take miscellaneous symbols, brackets, etc.)

Symbol.	Reference.	"A."	"B."	"C."
a_i, b_i	9.43			
C_{ij}, c_{ij}	9.33			
$e\alpha_m$	6.8	$e_{m-1}\,S\,(m+1)$		
g_m	1.6			
$g_{m-1,\,1}$	2.4			
$g_{s-u,\,u}$	3.7			
g_{-1}	3.7			
$h\gamma_m$	6.2	HM_m	HM_m	
$h\delta_m$	6.5	NH_{m-1}		
$h\delta_4 = \alpha_3 h$	6.9	N (O, T)		
I_m	8.9			
$(IA)_m$	8.1			
$(IA)_6 = 1_{22}$	11.8		V_{72}	
$(IA)_7 = 2_{22}$	11.8			
$\{k\}$	3.4	p_k	p_k	
$[\{k\}, \alpha_1]$	4.4	P_k	P_k	
$[\{k\}, \{k'\}]$	4.4	$(k\,;\ k')$		
k_u	3.4			k_u
$\{k_1, k_2, \ldots k_{m-1}\}$	3.4			$k_1 k_2 \ldots k_{m-1}$
n_{pq}	7.33			
$n_{(-1)\,q}$	7.77, 7.92			
$[n\ p\ q]$	7.6, 12.6			
O_{npq}	12.11			
P_m	8.3			
$(PA)_m$	8.1			
$(PA)_6 = 2_{21}$	11.6		V_{27}	
$(PA)_7 = 3_{21}$	11.5		V_{56}	
$(PA)_8 = 4_{21}$	11.7		V_{240}	
$(PA)_9 = 5_{21}$	9.1, 10.1			
${}_rR_m$, ${}_rR_n$	2.7			
R	9.1			
S	9.2			
S_m	8.9			
$(SA)_m$	8.1			
$(SA)_7 = 2_{31}$	9.6		V_{126}	
$(SA)_8 = 3_{31}$	9.6			
T	9.2, 10.8			
T_{ij}	9.2			
$T_{ijkl} = T_{ij}\,T_{kl}$	9.2			
$t_1(\ )\,;\ t_n(\ )$	5.1	$ce_1(\ )\,;\ ce_n(\ )$	$t(\ )\,;\ (\ )^n$	
$t_1 t_1(\ )\,;\ t_1 t_n(\ )$	5.9	$e_2(\ )\,;\ ce_{n-1}\,e_{n+1}(\ )$		
U_{ij}	9.1			
U_{ijk}, V_{ijk}	10.6			

Table of Symbols—(continued).

Symbol.	Reference.	"A."	"B."	"C."
$(\alpha_{p-1} \overline{}_{\sqrt{2}} \alpha_{q-1})$	4.5			
α_m	3.53	$S(m+1)$	S_m	
$\alpha_m h$	6.8			
α_3 ; α_4	3.5	T ; C_5	T ; C_5	33 ; 333
β_m	3.53	Cr_m	Cr_m	
β_3 ; β_4	3.5	O ; C_{16}	O ; C_{16}	34 ; 334
γ_m	3.53	M_m	M_m	
γ_3 ; γ_4	3.5	C ; C_8	C ; C_8	43 ; 433
δ_m	3.53	NM_{m-1}		
δ_4	4.9	NC		434
$\Delta_{m-u,u}$	2.8, 2.9			$K(u+1, m-1)$
ε_{pq}	4.73			
θ_m	2.8			R_{01}
$\theta_{m-u,u}$	2.8			θ_u
λ	4.71, 5.51			
Π_m	1.1			$(Po)_m$
Π_r	1.2			$(Po)_r$
Π_{-1}	1.3			
Π'_m, Π'_n	3.3			
$\Pi^{(p)}_{m_p}$	4.11			
Π^{+u}_r	7.1			
τ	3.61	$\frac{1}{2}(e+1)$	$\frac{1}{2}(e+1)$	$\frac{1}{2}(e+1)$
Ω	10.12			
1_{32}	7.45		V_{576}	
2_{41}	7.45		V_{2160}	
$\{2\}$	6.2			
$\{3, 4, 3\} = t_1\beta_4 = O_{111}$	12.1	C_{24}	C_{24}	343
$\{3, 3, 5\}$; $\{5, 3, 3\}$	3.5	C_{600} ; C_{120}	C_{600} ; C_{120}	335 ; 533
$\{3, 3, 4, 3\} = h\delta_5$	6.6	NC_{16}		3343
$\{3, 4, 3, 3\}$	3.5	NC_{24}		3433
$(1^p, 0^q)$	5.7			
$\pm(x_1, x_2, \ldots)'$	3.6	$[x_1, x_2, \ldots] : 2$	$[x_1, x_2, \ldots]\,\frac{1}{2}$	
$\left({}^{r}\middle\vert{}_{m}\right)$	1.2		R_r	N_r
$\left({}^{r}\middle\vert{}_{s}\right)$	1.2			N_{rs}
$\left({}^{s}_{r}\middle\vert{}_{m}\right)$	1.2			N_{sr}
$\left({}^{s}_{r}\middle\vert{}_{n}\right)$	1.2			${}_nN_{sr}$
$\left({}^{-1}\middle\vert{}_{m}\right)$, $\left({}^{s}_{-1}\middle\vert{}_{n}\right)$	1.3			
$\left({}^{s-1}\middle\vert{}_{m-1,1}\right)$	2.5			V_{s-1}
$\left({}^{s-u}\middle\vert{}_{m-u,u}\right)$	2.6			
$\left({}^{r}\middle\vert{}_{m}\right)'$	1.5			
$(\ )\times$	1.7			
$(\ )_n$	5.2			
$(\ )^{+u}$	7.1			
(ij)	9.2			
$[efgh \,.\, ijkl]$	9.34			
$[fgh \,.\, ijk]$	9.37			
$\left[\frac{m}{2}\right]$	6.2			

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